# A Rigorous Proof of the Capacity of MIMO Gauss-Markov Rayleigh Fading Channels 

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#### Abstract

We investigate the problem of message transmission over time-varying single-user multiple-input multiple-output (MIMO) Rayleigh fading channels with average power constraint and with complete channel state information available at the receiver side (CSIR). To describe the channel variations over the time, we consider a first-order Gauss-Markov model. We completely solve the problem by giving a single-letter characterization of the channel capacity in closed form and by providing a rigorous proof of $i t$.


## Index Terms

Gauss-Markov Rayleigh fading channels, channel capacity, multiple-antenna channels

## I. Introduction

In many new applications in modern wireless communications such as several machine-to-machine and human-to-machine systems, the tactile internet [1] and industry 4.0 [2], robust and ultra-reliable low latency information exchange is required. These applications impose challenges on the robustness requirement because of the time-varying nature of the channel conditions caused by the mobility and the changing wireless medium.

Several accurate tractable channel models are employed to model the channel variations appearing in wireless communications including the Markov model, often employed in flat fading and inter-symbol interference [3]. The Markov model is widely used for modeling wireless flat-fading channels due to its low memory and its consolidated theory.

The availability and quality of channel state information (CSI) has a high influence on the capacity of the Markov channels. Over the past decades, many researchers have addressed the problem of communication over finite-state Markov channels (FSMCs) [4] and extensive studies have been performed to analyze the capacity of FSMCs in closed form under the assumption of the availability of partial/complete channel state information at the sender and/or the receiver side [5]-[11].

In our work, the focus is on continuously time-varying Markov channels, which are of high relevance for practical systems. In particular, we are concerned with the time-varying single-user multiple-input multiple-output (MIMO) Rayleigh fading channels, where we assume that the statistics of the gain sequence are known to both the sender and the receiver and that the actual realization of the channel state sequence is completely known to the receiver only (CSIR). Therefore, the state sequence is viewed as a second output sequence of the channel. We further assume that the channel fades are modeled as a first-order Gauss-Markov process, which is widely used to describe the time-varying aspect of the channel [12]-[15]. The focus is on the multiple-antenna setting which has drawn considerable attention in the area of wireless communications because MIMO systems offer higher rates and more reliability and resistance to interference, compared to single-input single-output (SISO) systems [16].

To the best of our knowledge, no rigorous proof of the capacity of MIMO Gauss-Markov fading channels with CSIR is provided in the literature. A single-letter expression for the capacity is provided in [17] in the case when the channel fades are independent and identically distributed (i.i.d.). Other than that, only the proof of a general formula based on the inf-information rate for the capacity which can be generalized for arbitrary channels with abstract alphabets is provided in [18].

The main contribution of our work is to give a single-letter expression of the capacity of MIMO Gauss-Markov fading channels with average power constraint and to provide a rigorous proof of it.

Paper Outline: The rest of the paper is organized as follows. In Section II, we present the channel model, provide the key definitions and the main and auxiliary results. In Section III, we provide a rigorous proof of the capacity of time-varying multi-antenna Rayleigh fading channels with CSIR. Section IV is devoted to deriving an upper-bound on the variance of the

[^0]normalized information density between the inputs and the outputs of the time-varying MIMO Rayleigh fading channel. This auxiliary result is used in the proof of the capacity formula. Section V contains concluding remarks and proposes potential future research in this field. Several auxiliary lemmas are collected in the Appendix.

Notation: $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{R}$ denotes the set of real numbers; $H(\cdot)$ and $h(\cdot)$ correspond to the entropy and the differential entropy function, respectively; $I(\cdot ; \cdot)$ denotes the mutual information between two random variables. All information quantities are taken to base 2 . Throughout the paper, $\log$ is taken to base 2 . The natural exponential and the natural logarithm are denoted by exp and ln, respectively. For any random variables $X$ and $Y$ whose joint probability law has a density $p_{X, Y}(x, y)$, we denote their marginal probability density function by $p_{X}(x)$ and $p_{Y}(y)$, respectively, and their conditional probability density functions by $p_{X \mid Y}(x \mid y)$ and $p_{Y \mid X}(y \mid x)$. For any random variables $X, Y$ and $Z$, we use the notation $X \ominus Y \ominus Z$ to indicate a Markov chain. $|\mathcal{K}|$ stands for the cardinality of the set $\mathcal{K}$. tr refers to the trace operator. For any matrix $\mathbf{A},\|\mathbf{A}\|$ stands for the operator norm of $\mathbf{A}$ with respect to the Euclidean norm, $\mathbf{A}^{H}$ stands for the standard Hermitian transpose of $\mathbf{A}, \operatorname{vec}(\mathbf{A})$ refers to the vectorization of $\mathbf{A}, \lambda_{\max }(\mathbf{A})$ refers to the maximum eigenvalue of $\mathbf{A}$ and $\lambda_{\min }(\mathbf{A})$ refers to the its minimum eigenvalue. For any matrix $\mathbf{A}$ and $\mathbf{B}$, we use the notation $\mathbf{A} \preceq \mathbf{B}$ to indicate that $\mathbf{B}-\mathbf{A}$ is positive semi definite. For any vector $\boldsymbol{X}, \boldsymbol{X}^{T}$ refers to its transpose. For any random matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$ with entries $\mathbf{A}_{i, j}$ $i=1, \ldots, m, j=1, \ldots, n$, we define

$$
\mathbb{E}[\mathbf{A}]=\left[\begin{array}{ccc}
\mathbb{E}\left[\mathbf{A}_{11}\right] & \mathbb{E}\left[\mathbf{A}_{12}\right] & \cdots \\
\vdots & \ddots & \\
\mathbb{E}\left[\mathbf{A}_{m 1}\right] & & \mathbb{E}\left[\mathbf{A}_{m n}\right]
\end{array}\right]
$$

For any integer $m, \mathcal{Q}_{(P, m)}$ is defined to be the set of positive semi-definite Hermitian matrices which are elements of $\mathbb{C}^{m \times m}$ and whose trace is smaller than or equal to $P$.

## II. Channel Model, Definitions and Results

## A. Channel Model

For any block-length $n$, we consider the following channel model for the time-variant fading channel $W_{\mathbf{G}^{n}}$

$$
\begin{equation*}
\boldsymbol{z}_{i}=\mathbf{G}_{i} \boldsymbol{t}_{i}+\boldsymbol{\xi}_{i} \quad i=1 \ldots n \tag{1}
\end{equation*}
$$

where $\boldsymbol{t}^{n}=\left(\boldsymbol{t}_{1}, \ldots, \boldsymbol{t}_{n}\right) \in \mathbb{C}^{N_{T} \times n}$ and $\boldsymbol{z}^{n}=\left(\boldsymbol{z}_{1}, \ldots, \boldsymbol{z}_{n}\right) \in \mathbb{C}^{N_{R} \times n}$ are channel input and output blocks, respectively, and where $N_{T}$ and $N_{R}$ refer to the number of transmit and receive antennas, respectively.

Here, $\mathbf{G}^{n}=\mathbf{G}_{1} \ldots \mathbf{G}_{n}$, where $\mathbf{G}_{i}$ models the gain for the $i^{t h}$ channel use. We consider the following model for the gain. For $0 \leq \alpha<1$ :

$$
\begin{equation*}
\mathbf{G}_{i}=\sqrt{\alpha} \mathbf{G}_{i-1}+\sqrt{1-\alpha} \mathbf{W}_{i}, \quad i=2 \ldots n \tag{2}
\end{equation*}
$$

We assume that $\mathbf{G}_{1}$ and $\mathbf{W}_{i}, i=2 \ldots n$, are i.i.d., where $\mathbf{G}_{1}$ and $\mathbf{W}_{i}, i=2 \ldots n$, have i.i.d. entries and where $\operatorname{vec}\left(\mathbf{G}_{1}\right), \operatorname{vec}\left(\mathbf{W}_{i}\right), i=2 \ldots n$ are drawn from $\mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$. Therefore, the sequence of $\mathbf{G}_{i}, i=1 \ldots n$, forms a Markov chain. $\boldsymbol{\xi}^{n}=\left(\boldsymbol{\xi}_{1}, \ldots, \boldsymbol{\xi}_{n}\right) \in \mathbb{C}^{N_{R} \times n}$ models the noise sequence. We further assume that the $\boldsymbol{\xi}_{i} s$ are i.i.d., where $\boldsymbol{\xi}_{i} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R}}, \sigma^{2} \mathbf{I}_{N_{R}}\right), i=1 \ldots n$, that $\mathbf{G}^{n}$ and $\boldsymbol{\xi}^{n}$ are mutually independent and that $\left(\mathbf{G}^{n}, \boldsymbol{\xi}^{n}\right)$ is independent of the random input sequence $\boldsymbol{T}^{n}=\left(\boldsymbol{T}_{1}, \ldots, \boldsymbol{T}_{n}\right)$. It is also assumed that both the sender and the receiver know the statistics of the random gain sequence $\mathbf{G}^{n}$ and that only the receiver knows its actual realization (CSIR). Therefore, $\mathbf{G}^{n}$ is viewed as a second output sequence of the fading channel.

Remark 1. It follows from (2) that all fades are i.i.d. for $\alpha=0$. This scenario has been already treated in [17].

## B. Properties of the random gain sequence

In the following lemmas, we present some properties of the random gain in (2).
Lemma 1. For $0<\alpha<1$ and $i \in\{1 \ldots n\}$,

$$
\mathbf{G}_{i}=\sqrt{\alpha}^{i-1} \mathbf{G}_{1}+\sqrt{1-\alpha} \sum_{j=2}^{i} \sqrt{\alpha}^{i-j} \mathbf{W}_{j} .
$$

Proof. We will proceed by induction. Base Case: Clearly, the statement of the Lemma holds for $i=1$
Inductive step: Show that for any $k \geq 2$, if the statement of the lemma holds for $i=k$ then it holds for $i=k+1$.

Assume that the statement of the lemma holds for $i=k$, then we have

$$
\mathbf{G}_{k}=\sqrt{\alpha}^{k-1} \mathbf{G}_{1}+\sqrt{1-\alpha} \sum_{j=2}^{k} \sqrt{\alpha}^{k-j} \mathbf{W}_{j}
$$

It follows that

$$
\begin{aligned}
& \mathbf{G}_{k+1} \\
& \stackrel{(a)}{=} \sqrt{\alpha} \mathbf{G}_{k}+\sqrt{1-\alpha} \mathbf{W}_{k+1} \\
& \stackrel{(b)}{=} \sqrt{\alpha}\left[\sqrt{\alpha}^{k-1} \mathbf{G}_{1}+\sqrt{1-\alpha} \sum_{j=2}^{k} \sqrt{\alpha}^{k-j} \mathbf{W}_{j}\right]+\sqrt{1-\alpha} \mathbf{W}_{k+1} \\
& =\sqrt{\alpha}^{k} \mathbf{G}_{1}+\sqrt{1-\alpha} \sum_{j=2}^{k} \sqrt{\alpha}^{k+1-j} \mathbf{W}_{j}+\sqrt{1-\alpha} \mathbf{W}_{k+1} \\
& =\sqrt{\alpha}^{k} \mathbf{G}_{1}+\sqrt{1-\alpha} \sum_{j=2}^{k} \sqrt{\alpha}^{k+1-j} \mathbf{W}_{j}+\sqrt{1-\alpha} \sqrt{\alpha}^{k+1-(k+1)} \mathbf{W}_{k+1} \\
& =\sqrt{\alpha}^{k} \mathbf{G}_{1}+\sqrt{1-\alpha} \sum_{j=2}^{k+1} \sqrt{\alpha}^{k+1-j} \mathbf{W}_{j}
\end{aligned}
$$

where $(a)$ follows from (2) and $(b)$ follows from the induction assumption. Thus, the statement of the lemma holds for $i=k+1$.
Conclusion: Since both the base case and the inductive step have been proved as true, by mathematical induction the statement of the lemma holds for every $i=1 \ldots n$.

Lemma 2. $\forall i \in\{1, \ldots, n\}$, it holds that

$$
\operatorname{vec}\left(\mathbf{G}_{i}\right) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)
$$

where $\mathbf{G}_{i}, i=1 \ldots n$ is defined in (2) with $0 \leq \alpha<1$.

Proof. Clearly, the statement of the lemma holds for $\alpha=0$. Now, let $0<\alpha<1$. The statement of the lemma is valid for $i=1$. Let $i \in\{2, \ldots, n\}$ be fixed arbitrarily. Let $\mathbf{G}_{1}^{\prime}=\sqrt{\alpha}^{i-1} \mathbf{G}_{1}$ and $\mathbf{W}_{j}^{\prime}=\sqrt{1-\alpha} \sqrt{\alpha}{ }^{i-j} \mathbf{W}_{j}$ for every $j \in\{2, \ldots, i\}$. Since $\mathbf{G}_{1}$ and $\mathbf{W}_{j}, j=2, \ldots, n$ are independent, it follows that $\mathbf{G}_{1}^{\prime}$ and $\boldsymbol{W}_{j}^{\prime}, j=2, \ldots, n$ are also independent. Since $\operatorname{vec}\left(\mathbf{G}_{1}\right) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$ and $\operatorname{vec}\left(\mathbf{W}_{j}\right) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$ for every $j \in\{2, \ldots i\}$, it follows that

$$
\operatorname{vec}\left(\mathbf{G}_{1}^{\prime}\right) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \alpha^{i-1} \mathbf{I}_{N_{R} N_{T}}\right)
$$

and that for every $j \in\{2, \ldots, i\}$

$$
\operatorname{vec}\left(\boldsymbol{W}_{j}^{\prime}\right) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}},(1-\alpha) \alpha^{i-j} \mathbf{I}_{N_{R} N_{T}}\right)
$$

Now, from Lemma 1, it follows that

$$
\mathbf{G}_{i}=\mathbf{G}_{1}^{\prime}+\sum_{j=2}^{i} \mathbf{W}_{j}^{\prime}
$$

As a result,

$$
\operatorname{vec}\left(\mathbf{G}_{i}\right) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}},\left[\alpha^{i-1}+(1-\alpha) \sum_{j=2}^{i} \alpha^{i-j}\right] \mathbf{I}_{N_{R} N_{T}}\right)
$$

For $0<\alpha<1$, we have

$$
\begin{aligned}
\sum_{j=2}^{i} \alpha^{i-j} & =\alpha^{i} \sum_{j=2}^{i}\left(\frac{1}{\alpha}\right)^{j} \\
& =\alpha^{i}\left(\frac{1}{\alpha}\right)^{2} \frac{1-\left(\frac{1}{\alpha}\right)^{i-1}}{1-\frac{1}{\alpha}} \\
& =\frac{\alpha^{i}-\alpha}{\alpha^{2}-\alpha} \\
& =\frac{1-\alpha^{i-1}}{1-\alpha}
\end{aligned}
$$

It follows that

$$
\alpha^{i-1}+(1-\alpha) \sum_{j=2}^{i} \alpha^{i-j}=1
$$

This yields

$$
\operatorname{vec}\left(\mathbf{G}_{i}\right) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right) \quad \forall i \in\{1, \ldots n\}
$$

Lemma 3. Let $i_{1}, i_{2} \in\{1, \ldots, n\}$. Assume without loss of generality that $i_{1}<i_{2}$. We consider the gain model presented in (2). Then, for $0<\alpha<1$, it holds that

$$
\mathbf{G}_{i_{2}}=\sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}}+\sqrt{1-\alpha} \sum_{j=i_{1}+1}^{i_{2}} \sqrt{\alpha}^{i_{2}-j} \mathbf{W}_{j}
$$

Proof. By Lemma 1, it holds that

$$
\mathbf{G}_{i_{2}}=\sqrt{\alpha}^{i_{2}-1} \mathbf{G}_{1}+\sqrt{1-\alpha} \sum_{j=2}^{i_{2}} \sqrt{\alpha}^{i_{2}-j} \mathbf{W}_{j}
$$

and that

$$
\mathbf{G}_{i_{1}}=\sqrt{\alpha}^{i_{1}-1} \mathbf{G}_{1}+\sqrt{1-\alpha} \sum_{j=2}^{i_{1}} \sqrt{\alpha}^{i_{1}-j} \mathbf{W}_{j}
$$

Thus

$$
\begin{aligned}
& \mathbf{G}_{i_{2}}-\sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \\
& =\sqrt{\alpha}^{i_{2}-1} \mathbf{G}_{1}+\sqrt{1-\alpha} \sum_{j=2}^{i_{2}} \sqrt{\alpha}^{i_{2}-j} \mathbf{W}_{j}-\sqrt{\alpha}^{i_{2}-i_{1}}\left[\sqrt{\alpha}^{i_{1}-1} \mathbf{G}_{1}+\sqrt{1-\alpha} \sum_{j=2}^{i_{1}} \sqrt{\alpha}^{i_{1}-j} \mathbf{W}_{j}\right] \\
& =\sqrt{\alpha}^{i_{2}-1} \mathbf{G}_{1}+\sqrt{1-\alpha} \sum_{j=2}^{i_{2}} \sqrt{\alpha}^{i_{2}-j} \mathbf{W}_{j}-\sqrt{\alpha}^{i_{2}-1} \mathbf{G}_{1}-\sqrt{1-\alpha} \sum_{j=2}^{i_{1}} \sqrt{\alpha}^{i_{2}-j} \mathbf{W}_{j} \\
& =\sqrt{1-\alpha} \sum_{j=i_{1}+1}^{i_{2}} \sqrt{\alpha}^{i_{2}-j} \mathbf{W}_{j} .
\end{aligned}
$$

## C. Achievable Rate and Capacity

Next, we define an achievable rate for the channel $W_{\mathbf{G}^{n}}$ and the corresponding capacity. For this purpose, we begin by providing the definition of a transmission-code for $W_{\mathbf{G}^{n}}$.
Definition 1. A transmission-code $\Gamma$ of length $n$ and size ${ }^{1}\|\Gamma\|$ with average power constraint $P$ for the channel $W_{\mathbf{G}^{n}}$ is a family of pairs $\left\{\left(\mathbf{t}_{\ell}, \mathcal{D}_{\ell}^{\left(\mathbf{g}^{n}\right)}\right): \operatorname{vec}(\mathbf{g})^{n} \in \mathbb{C}^{N_{R} N_{T} \times n}, \quad \ell=1, \ldots,\|\Gamma\|\right\}$ such that for all $\ell, j \in\{1, \ldots,\|\Gamma\|\}$ and all $\mathbf{g}^{n}$ for which $\operatorname{vec}(\mathbf{g})^{n} \in \mathbb{C}^{N_{R} N_{T} \times n}$, we have:

$$
\begin{align*}
& \mathbf{t}_{\ell} \in \mathbb{C}^{N_{T} \times n}, \quad \mathcal{D}_{\ell}^{\left(\mathbf{g}^{n}\right)} \subset \mathbb{C}^{N_{R} \times n} \\
& \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{t}_{\ell, i}^{H} \boldsymbol{t}_{\ell, i} \leq P \quad \mathbf{t}_{\ell}=\left(\boldsymbol{t}_{\ell, 1}, \ldots, \boldsymbol{t}_{\ell, n}\right)  \tag{3}\\
& \mathcal{D}_{\ell}^{\left(\mathbf{g}^{n}\right)} \cap \mathcal{D}_{j}^{\left(\mathbf{g}^{n}\right)}=\varnothing, \quad \ell \neq j
\end{align*}
$$

Here, $\boldsymbol{t}_{\ell}, \ell=1, \ldots,\|\Gamma\|$ and $\mathcal{D}_{\ell}^{\left(\mathrm{g}^{n}\right)}, \ell=1, \ldots,\|\Gamma\|$, are the codewords and the decoding regions, respectively.
Definition 2. A real number $R$ is called an achievable rate of the channel $W_{\mathbf{G}^{n}}$ if for every $\theta, \delta>0$ there exists a code sequence $\left(\Gamma_{n}\right)_{n=1}^{\infty}$, where each code $\Gamma_{n}$ of length $n$ is defined according to Definition 1, such that

$$
\frac{\log \left\|\Gamma_{n}\right\|}{n} \geq R-\delta
$$

and

$$
e_{\max }\left(\Gamma_{n}\right)=\max _{\ell \in\left\{1 \ldots\left\|\Gamma_{n}\right\|\right\}} \mathbb{E}\left[W_{\mathbf{G}^{n}}\left(\mathcal{D}_{\ell}^{\left(\mathbf{G}^{n}\right)_{c}} \mid \boldsymbol{t}_{\ell}\right)\right] \leq \theta
$$

for sufficiently large $n$.
Definition 3. The supremum of all achievable rates defined according to Definition 2 is called the capacity of the fading channel $W_{\mathbf{G}^{n}}$ and is denoted by $C\left(P, N_{R} \times N_{T}\right)$.

## D. Main Result

In this section, we present the main result of our work, which is a single-letter characterization of the time-varying MIMO Gauss-Markov Rayleigh fading channel. This is illustrated in the following theorem.
Theorem 1. Let $\mathbf{G}$ be any random matrix with i.i.d. entries such that $\operatorname{vec}(\mathbf{G}) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$. A single-letter characterization of the capacity of the channel in (1) with gain model in (2) with $0 \leq \alpha<1$ is

$$
C\left(P, N_{R} \times N_{T}\right)=\max _{\mathbf{Q} \in \mathcal{Q}_{\left(P, N_{T}\right)}} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G Q G}^{H}\right)\right]
$$

The proof of Theorem 1 is provided in Section III.

## E. Auxiliary Result

For the proof of Theorem 1, we require the following auxiliary result on the normalized information density of $W_{\mathbf{G}^{n}}$.
Lemma 4. Let $\boldsymbol{T}^{n}=\left(\boldsymbol{T}_{1}, \ldots, \boldsymbol{T}_{n}\right)$ be an n-length input sequence of the channel $W_{\mathbf{G}^{n}}$ in (1) with gain model in (2) such that $0<\alpha<1$ and such that the $\boldsymbol{T}_{i}$ s are i.i.d., where $\boldsymbol{T}_{i} \sim \mathcal{N}\left(\mathbf{0}_{N_{T}}, \tilde{\mathbf{Q}}\right), i=1 \ldots n$, and $\tilde{\mathbf{Q}} \in \mathcal{Q}_{\left(P, N_{T}\right)}$. Let $\boldsymbol{Z}^{n}=\left(\boldsymbol{Z}_{1}, \ldots, \boldsymbol{Z}_{n}\right)$ be the corresponding output sequence. Then, it holds that

$$
\operatorname{var}\left(\frac{i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)}{n}\right) \leq \kappa(n)
$$

where $\kappa(n)=\frac{2 c^{\prime}}{n(1-\sqrt{\alpha})}+\frac{c^{\prime \prime}}{n}$ for some $c^{\prime}, c^{\prime \prime}>0$ and where $\lim _{n \rightarrow \infty} \kappa(n)=0$.
The proof of Lemma 4 is provided in Section IV.

## III. Proof of Theorem 1

The result of Theorem 1 is well-known for $\alpha=0$ [17]. The proof is then restricted for $0<\alpha<1$.
${ }^{1}$ This is the same notation used in [19].

## A. Direct Proof

Let

$$
R_{\max }=\max _{\mathbf{Q} \in \mathcal{Q}_{\left(P, N_{T}\right)}} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G} \mathbf{Q} \mathbf{G}^{H}\right)\right]
$$

where $\mathbf{G} \in \mathbb{C}^{N_{R} \times N_{T}}$ is any random matrix with i.i.d. entries such that $\operatorname{vec}(\mathbf{G}) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$. We are going to show that

$$
C\left(P, N_{R} \times N_{T}\right) \geq R_{\max }-\epsilon
$$

with $\epsilon$ being an arbitrarily small positive constant. Let $\theta, \delta>0$ and

$$
E_{n}=\left\{\boldsymbol{t}^{n}=\left(\boldsymbol{t}_{1}, \ldots, \boldsymbol{t}_{n}\right) \in \mathbb{C}^{N_{T} \times n}: \frac{1}{n} \sum_{i=1}^{n}\left\|\boldsymbol{t}_{i}\right\|^{2} \leq P\right\}
$$

We define for any $\mathbf{Q} \in \mathcal{Q}_{\left(P, N_{T}\right)}$,

$$
\phi(\mathbf{Q})=\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G} \mathbf{Q} \mathbf{G}^{H}\right)\right]
$$

Now notice that any $\mathbf{Q} \in Q_{\left(P, N_{T}\right)}$, we have

$$
\begin{aligned}
& \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G Q G}{ }^{H}\right) \\
& \stackrel{(a)}{\leq} \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}}\left\|\mathbf{G Q G} \mathbf{G}^{H}\right\| \mathbf{I}_{N_{R}}\right) \\
& =\log \operatorname{det}\left(\left[1+\frac{1}{\sigma^{2}}\left\|\mathbf{G Q G} \mathbf{G}^{H}\right\|\right] \mathbf{I}_{N_{R}}\right) \\
& =N_{R} \log \left(1+\frac{1}{\sigma^{2}}\left\|\mathbf{G Q G}{ }^{H}\right\|\right) \\
& \leq \frac{N_{R}}{\ln (2) \sigma^{2}}\left\|\mathbf{G} \mathbf{Q G}{ }^{H}\right\| \\
& \leq \frac{N_{R}}{\ln (2) \sigma^{2}}\|\mathbf{Q}\|\|\mathbf{G}\|^{2} \\
& (b) \frac{P N_{R}}{\leq}\|\mathbf{G}\|^{2} \\
& \ln (2) \sigma^{2}
\end{aligned}
$$

where (a) follows because $\mathbf{A} \preceq\|\mathbf{A}\| \mathbf{I}_{n}$ for any Hermitian $\mathbf{A} \in \mathbb{C}^{n \times n}$ (by Lemma 10 in the Appendix) and (b) follows because $\|\mathbf{Q}\|=\lambda_{\max }(\mathbf{Q}) \leq \operatorname{tr}(\mathbf{Q}) \leq P$. Now, it holds that $\mathbb{E}\left[\frac{P N_{R}}{\ln (2) \sigma^{2}}\|\mathbf{G}\|^{2}\right]<\infty$ since $\mathbb{E}\left[\|\mathbf{G}\|^{2}\right]<\infty$ (from Lemma 13 in the Appendix). Therefore, it follows from the dominated convergence theorem that $\phi$ is continuous on the compact set $\mathcal{Q}_{\left(P, N_{T}\right)}$. Therefore, one can find a $\tilde{\mathbf{Q}} \in \mathcal{Q}_{\left(P, N_{T}\right)}$ such that $\operatorname{tr}(\tilde{\mathbf{Q}})=P-\beta$ for some $\beta>0$ and such that

$$
\begin{equation*}
\phi(\tilde{\mathbf{Q}}) \geq R_{\max }-\epsilon \tag{4}
\end{equation*}
$$

We define

$$
\hat{P}=P-\beta
$$

and

$$
\begin{equation*}
\hat{\beta}=\frac{\beta}{\ln (2) \hat{P}}-\log \left(1+\frac{\beta}{\hat{P}}\right)>0 \tag{5}
\end{equation*}
$$

Let us now introduce the following well-known lemma:
Lemma 5. (Feinstein's Lemma with input constraints) [20] Let $n>0$ be fixed arbitrarily. Consider any channel with random input sequence $T^{n}$, with corresponding random channel output sequence $Z^{n}$ and with information density $i\left(T^{n} ; Z^{n}\right)$. Then, for any integer $\tau>0$, real number $\gamma>0$, and measurable set $E_{n}$, there exists a code with cardinality $\tau$, maximum error probability $\epsilon_{n}$ and block-length $n$, whose codewords are contained in the set $E_{n}$, where $\epsilon_{n}$ satisfies

$$
\epsilon_{n} \leq \mathbb{P}\left[\frac{1}{n} i\left(T^{n} ; Z^{n}\right) \leq \frac{\log \tau}{n}+\gamma\right]+\mathbb{P}\left[T^{n} \notin E_{n}\right]+2^{-n \gamma}
$$

Let $\boldsymbol{T}^{n}=\left(\boldsymbol{T}_{1}, \ldots, \boldsymbol{T}_{n}\right) \in \mathbb{C}^{N_{T} \times n}$ to be the random input sequence of the channel $W_{\mathbf{G}^{n}}$, where the $\boldsymbol{T}_{i} s$ are i.i.d. such that $\boldsymbol{T}_{i} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{T}}, \tilde{\mathbf{Q}}\right), i=1 \ldots n$. We denote its corresponding random output sequence by $\boldsymbol{Z}^{n}=\left(\boldsymbol{Z}_{1}, \ldots, \boldsymbol{Z}_{n}\right)$. Now, we apply Lemma 5 for $E_{n}=\left\{\boldsymbol{t}^{n}=\left(\boldsymbol{t}_{1}, \ldots, \boldsymbol{t}_{n}\right) \in \mathbb{C}^{N_{T} \times n}: \frac{1}{n} \sum_{i=1}^{n}\left\|\boldsymbol{t}_{i}\right\|^{2} \leq P\right\}$ and for $\gamma=\frac{\delta}{4}$. It follows that there exists a code sequence $\left(\Gamma_{n}\right)_{n=1}^{\infty}$, where each code $\Gamma_{n}$ is defined according to Definition 1 such that

$$
\begin{equation*}
e_{\max }\left(\Gamma_{n}\right) \leq \mathbb{P}\left[\frac{1}{n} i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) \leq \frac{1}{n} \log \left\|\Gamma_{n}\right\|+\frac{\delta}{4}\right]+\mathbb{P}\left[\boldsymbol{T}^{n} \notin E_{n}\right]+2^{-n \frac{\delta}{4}} \tag{6}
\end{equation*}
$$

where here $\mathbf{G}^{n}$ is viewed as a second output sequence of $W_{\mathbf{G}^{n}}$ because we assume CSIR and where

$$
\begin{aligned}
e_{\max }\left(\Gamma_{n}\right) & =\max _{\ell \in\left\{1 \ldots \mid \Gamma_{n} \|\right\}} \mathbb{E}\left[W_{\mathbf{G}^{n}}\left(\mathcal{D}_{\ell}^{\left(\mathbf{G}^{n}\right)_{c}} \mid \boldsymbol{t}_{\ell}\right)\right] \\
& =\max _{\ell \in\{1 \ldots|\mathcal{M}|\}} \mathbb{E}\left[\mathbb{P}\left[\hat{M} \neq \ell \mid M=\ell, \mathbf{G}^{n}\right]\right] \\
& =\max _{\ell \in\{1 \ldots|\mathcal{M}|\}} \mathbb{P}[\hat{M} \neq \ell \mid M=\ell]
\end{aligned}
$$

with $M, \hat{M}$ being the random message and the random decoded message and with $\mathcal{M}$ being the set of messages.
Choose $\left\|\Gamma_{n}\right\|$ such that for sufficiently large $n$

$$
R_{\max }-\epsilon-\delta \leq \frac{\log \left\|\Gamma_{n}\right\|}{n} \leq R_{\max }-\epsilon-\frac{\delta}{2}
$$

It follows that

$$
\begin{align*}
e_{\max }\left(\Gamma_{n}\right) & \leq \mathbb{P}\left[\frac{1}{n} i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) \leq R_{\max }-\epsilon-\frac{\delta}{2}\right]+\mathbb{P}\left[\boldsymbol{T}^{n} \notin E_{n}\right]+2^{-n \frac{\delta}{4}} \\
& \leq \mathbb{P}\left[\frac{1}{n} i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) \leq \phi(\tilde{\mathbf{Q}})-\frac{\delta}{2}\right]+\mathbb{P}\left[\boldsymbol{T}^{n} \notin E_{n}\right]+2^{-n \frac{\delta}{4}} \tag{7}
\end{align*}
$$

where we used (4) in the last step. It remains to find upper-bounds for $\mathbb{P}\left[\frac{1}{n} i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) \leq \phi(\tilde{\mathbf{Q}})-\frac{\delta}{2}\right]$ and for $\mathbb{P}\left[\boldsymbol{T}^{n} \notin E_{n}\right]$ that vanish as $n$ goes to infinity.

1) Upper-bound for $\mathbb{P}\left[\boldsymbol{T}^{n} \notin E_{n}\right]$ : We will prove that

$$
\mathbb{P}\left[\boldsymbol{T}^{n} \notin E_{n}\right] \leq 2^{-n \hat{\beta}}
$$

where $\hat{\beta}$ is defined in (5). For this purpose, we will introduce and prove the following lemma:
Lemma 6. Let $\boldsymbol{X}_{i}, i=1, \ldots, n$ be i.i.d. $N$-dimensional complex Gaussian random vectors with mean $\mathbf{0}_{N}$ and covariance matrix $\mathbf{O}$ whose trace is smaller than or equal to $\rho$. Then, for any $\delta>0$

$$
\mathbb{P}\left[\sum_{i=1}^{n}\left\|\boldsymbol{X}_{i}\right\|^{2} \geq n(\rho+\delta)\right] \leq\left[\left(1+\frac{\delta}{\rho}\right) 2^{-\frac{\delta}{\ln (2) \rho}}\right]^{n}
$$

where

$$
\left\|\boldsymbol{X}_{i}\right\|^{2}=\sum_{j=1}^{N}\left|\boldsymbol{X}_{i}^{j}\right|^{2}
$$

and

$$
\boldsymbol{X}_{i}=\left(\boldsymbol{X}_{i}^{1}, \ldots, \boldsymbol{X}_{i}^{N}\right)^{T}
$$

Proof. Let $\boldsymbol{X}$ be a random vector with the same distribution as each of the $\boldsymbol{X}_{i}$. Then

$$
\begin{align*}
& \mathbb{P}\left[\sum_{i=1}^{n}\left\|\boldsymbol{X}_{i}\right\|^{2} \geq n(\rho+\delta)\right] \\
& =\mathbb{P}\left[\sum_{i=1}^{n}\left\|\boldsymbol{X}_{i}\right\|^{2}-n(\rho+\delta) \geq 0\right] \\
& \leq \mathbb{E}\left[\exp \left(\beta\left(\sum_{i=1}^{n}\left\|\boldsymbol{X}_{i}\right\|^{2}-n(\rho+\delta)\right)\right]\right. \\
& =\left[\exp (-[\rho+\delta] \beta) \mathbb{E}\left[\exp \left(\beta\|\boldsymbol{X}\|^{2}\right]\right]^{n}\right. \tag{8}
\end{align*}
$$

where we used the $\boldsymbol{X}_{i} s$ are i.i.d.. By a standard calculation which follows below, one can show that

$$
\begin{aligned}
\mathbb{E}\left[\exp \left(\beta\|\boldsymbol{X}\|^{2}\right)\right] & =\mathbb{E}\left[\exp \left(\beta \boldsymbol{X}^{H} \boldsymbol{X}\right)\right] \\
& =\prod_{j=1}^{N}\left(1-\beta \mu_{j}\right)^{-1} \quad \beta<\beta_{0},
\end{aligned}
$$

where $\mu_{1}, \ldots, \mu_{N}$ are the eigenvalues of $\mathbf{O}$, and for $\beta_{0}=\frac{1}{\rho} \leq \frac{1}{\mu_{1}+\ldots+\mu_{N}} \leq \min _{j \in\{1, \ldots, N\}} \frac{1}{\mu_{j}}$ so that all the factors are positive, whether $\mathbf{O}$ is non-singular or singular. To prove this, we let $r$ be the rank of $\mathbf{O}$. It holds that $r \leq N$. We make use of the spectral decomposition theorem to express $\mathbf{O}$ as $\mathbf{S}_{\mathbf{O}}^{\star} \Lambda^{\star} S_{\mathbf{O}}^{\star}{ }^{H}$, where $\Lambda^{\star}$ is a diagonal matrix whose first $r$ diagonal elements are positive and where the remaining diagonal elements are equal to zero. Next, we let $\mathbf{V}^{\star}=\mathbf{S}_{\mathrm{O}}^{\star} \Lambda^{\star \frac{1}{2}}$ and remove the $N-r$ last columns of $\mathbf{V}^{\star}$, which are null vectors to obtain the matrix $\mathbf{V}$. Then, it can be verified that $\mathbf{O}=\mathbf{V} \mathbf{V}^{H}$. We can write $\boldsymbol{X}=\mathbf{V} \boldsymbol{U}^{\star}$ where $\boldsymbol{U}^{\star} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \mathbf{I}_{r}\right)$. As a result:

$$
\boldsymbol{X}^{H} \boldsymbol{X}=\left(\boldsymbol{U}^{\star}\right)^{H} \mathbf{V}^{H} \mathbf{V} \boldsymbol{U}^{\star} .
$$

Let $\mathbf{S}$ be a unitary matrix which diagonalizes $\mathbf{V}^{H} \mathbf{V}$ such that $\mathbf{S}^{H} \mathbf{V}^{H} \mathbf{V} \mathbf{S}=\operatorname{Diag}\left(\mu_{1}, \ldots, \mu_{r}\right)$ with $\mu_{1}, \ldots, \mu_{r}$ being the positive eigenvalues of $\mathbf{O}=\mathbf{V} \mathbf{V}^{H}$ in decreasing order. One defines $\boldsymbol{U}=\mathbf{S}^{H} \boldsymbol{U}^{\star}$. We have

$$
\begin{aligned}
\operatorname{cov}(\boldsymbol{U}) & =\mathbf{S}^{H} \operatorname{cov}\left(\boldsymbol{U}^{\star}\right) \mathbf{S} \\
& =\mathbf{S}^{H} \mathbf{S} \\
& =\mathbf{I}_{r} .
\end{aligned}
$$

Therefore, it holds that $\boldsymbol{U} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \mathbf{I}_{r}\right)$. Since $\mathbf{S}$ is unitary, we have

$$
\begin{aligned}
\boldsymbol{X}^{H} \boldsymbol{X} & =\left(\left(\mathbf{S}^{H}\right)^{-1} \boldsymbol{U}\right)^{H} \mathbf{V}^{H} \mathbf{V}\left(\mathbf{S}^{H}\right)^{-1} \boldsymbol{U} \\
& =\boldsymbol{U}^{H} \mathbf{S}^{H} \mathbf{V}^{H} \mathbf{V} \mathbf{S} \boldsymbol{U} \\
& =\boldsymbol{U}^{H} \operatorname{Diag}\left(\mu_{1}, \ldots, \mu_{r}\right) \boldsymbol{U} \\
& =\sum_{j=1}^{r} \mu_{j}\left|\boldsymbol{U}_{j}\right|^{2} .
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\mathbb{E}\left[\exp \left(\beta\|\boldsymbol{X}\|^{2}\right)\right] & =\mathbb{E}\left[\prod_{j=1}^{r} \exp \left(\frac{1}{2} \beta \mu_{j} 2\left|\boldsymbol{U}_{j}\right|^{2}\right)\right] \\
& =\prod_{j=1}^{r} \mathbb{E}\left[\exp \left(\frac{1}{2} \beta \mu_{j} 2\left|\boldsymbol{U}_{j}\right|^{2}\right)\right] \\
& =\prod_{j=1}^{N}\left(1-\beta \mu_{j}\right)^{-1},
\end{aligned}
$$

where we used that all the $\boldsymbol{U}_{j}$ 's are independent, that $\forall j \in\{1, \ldots, r\}, 2\left|\boldsymbol{U}_{j}\right|^{2}$ is chi-square distributed with $k=2$ degrees of freedom and with moment generating function equal to $\mathbb{E}\left[\exp \left(2 t\left|\boldsymbol{U}_{j}\right|^{2}\right)\right]=(1-2 t)^{-k / 2}$ for $t<\frac{1}{2}$ and that $\forall j \in\{1, \ldots, r\}$ and for $\beta<\beta_{0}, \frac{1}{2} \beta \mu_{j}<\frac{1}{2}$. This completes the standard calculation.

Now, it holds that

$$
\prod_{i=1}^{N}\left(1-\beta \mu_{i}\right) \geq 1-\beta\left(\mu_{1}+\ldots+\mu_{N}\right) \geq 1-\beta \rho .
$$

This yields

$$
\exp (-(\rho+\delta) \beta) \mathbb{E}\left[\exp \left(\beta\|\boldsymbol{X}\|^{2}\right] \leq \frac{\exp (-(\rho+\delta) \beta)}{1-\beta \rho}\right.
$$

where $0<\beta<\frac{1}{\rho}=\beta_{0}$. Putting $\beta=\frac{\delta}{\rho(\delta+\rho)}<\frac{1}{\rho}$ yields

$$
\begin{aligned}
\exp (-(\rho+\delta) \beta) \mathbb{E}\left[\exp \left(\beta\|\boldsymbol{X}\|^{2}\right)\right] & \leq\left(1+\frac{\delta}{\rho}\right) \exp \left(-\frac{\delta}{\rho}\right) \\
& =\left(1+\frac{\delta}{\rho}\right) 2^{\left(-\frac{\delta}{\ln (2) \rho}\right)}
\end{aligned}
$$

which combined with (8) proves the lemma.
By Lemma 6, it holds that

$$
\begin{aligned}
\mathbb{P}\left[\sum_{i=1}^{n}\left\|\boldsymbol{T}_{i}\right\|^{2} \geq n(\hat{P}+\beta)\right] & \leq\left[\left(1+\frac{\beta}{\hat{P}}\right) 2^{\left(-\frac{\beta}{\ln (2) P}\right)}\right]^{n} \\
& =2^{\left(-n \frac{\beta}{\ln (2) P}+n \log \left(1+\frac{\beta}{P}\right)\right)} \\
& =2^{-n \hat{\beta}}
\end{aligned}
$$

As a result, we have

$$
\begin{align*}
\mathbb{P}\left[\boldsymbol{T}^{n} \notin E_{n}\right] & =\mathbb{P}\left[\sum_{i=1}^{n}\left\|\boldsymbol{T}_{i}\right\|^{2}>n P\right] \\
& \leq \mathbb{P}\left[\sum_{i=1}^{n}\left\|\boldsymbol{T}_{i}\right\|^{2} \geq n(\hat{P}+\beta)\right] \\
& \leq 2^{-n \hat{\beta}} \tag{9}
\end{align*}
$$

2) Upper-bound for $\mathbb{P}\left[\frac{1}{n} i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) \leq \phi(\tilde{\mathbf{Q}})-\frac{\delta}{2}\right]$ : Let us introduce the following lemma:

## Lemma 7.

$$
i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)=\sum_{i=1}^{n} i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)
$$

Proof. We have

$$
\begin{aligned}
i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) & =\log \left(\frac{p_{\boldsymbol{T}^{n}, \boldsymbol{Z}^{n}, \mathbf{G}^{n}}\left(\boldsymbol{T}^{n}, \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)}{p_{\boldsymbol{Z}^{n}, \mathbf{G}^{n}}\left(\boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) p_{\boldsymbol{T}^{n}}\left(\boldsymbol{T}^{n}\right)}\right) \\
& =\log \left(\frac{p_{\boldsymbol{Z}^{n}, \mathbf{G}^{n} \mid \boldsymbol{T}^{n}}\left(\boldsymbol{Z}^{n}, \mathbf{G}^{n} \mid \boldsymbol{T}^{n}\right)}{p_{\boldsymbol{Z}^{n}, \mathbf{G}^{n}}\left(\boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)}\right)
\end{aligned}
$$

Since $\mathbf{G}^{n}$ and $\boldsymbol{T}^{n}$ are independent, we have

$$
\log \left(\frac{p_{\boldsymbol{Z}^{n}, \mathbf{G}^{n} \mid \boldsymbol{T}^{n}}\left(\boldsymbol{Z}^{n}, \mathbf{G}^{n} \mid \boldsymbol{T}^{n}\right)}{p_{\boldsymbol{Z}^{n}, \mathbf{G}^{n}}\left(\boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)}\right)=\log \left(\frac{p_{\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}, \boldsymbol{T}^{n}}\left(\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}, \boldsymbol{T}^{n}\right)}{p_{\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}}\left(\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}\right)}\right)
$$

Furthermore, since conditioned on $\left(\mathbf{G}^{n}, \boldsymbol{T}^{n}\right)$, the outputs are independent, we have

$$
\log \left(\frac{p_{\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}, \boldsymbol{T}^{n}}\left(\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}, \boldsymbol{T}^{n}\right)}{p_{\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}}\left(\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}\right)}\right)=\log \left(\frac{\prod_{i=1}^{n} p_{\boldsymbol{Z}_{i} \mid \mathbf{G}^{n}, \boldsymbol{T}^{n}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}^{n}, \boldsymbol{T}^{n}\right)}{p_{\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}}\left(\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}\right)}\right)
$$

This yields

$$
\begin{aligned}
& i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) \\
& =\log \left(\frac{\prod_{i=1}^{n} p_{\boldsymbol{Z}_{i} \mid \mathbf{G}^{n}, \boldsymbol{T}^{n}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}^{n}, \boldsymbol{T}^{n}\right)}{p_{\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}}\left(\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}\right)}\right) \\
& \stackrel{(a)}{=} \log \left(\frac{\prod_{i=1}^{n} p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{p_{\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}}\left(\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}\right)}\right)
\end{aligned}
$$

where (a) follows because

$$
\mathbf{G}_{1} \boldsymbol{T}_{1} \ldots \mathbf{G}_{i-1} \boldsymbol{T}_{i-1} \mathbf{G}_{i+1} \boldsymbol{T}_{i+1} \ldots \mathbf{G}_{n} \boldsymbol{T}_{n} \boldsymbol{Z}^{i-1} \ominus \mathbf{G}_{i} \boldsymbol{T}_{i} \ominus \boldsymbol{Z}_{i}
$$

forms a Markov chain.
Now since conditioned on $\mathbf{G}^{n}$ and for independent inputs, the outputs are independent, we have

$$
\begin{aligned}
& \log \left(\frac{\prod_{i=1}^{n} p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{p_{\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}}\left(\boldsymbol{Z}^{n} \mid \mathbf{G}^{n}\right)}\right) \\
& =\log \left(\frac{\prod_{i=1}^{n} p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{\prod_{i=1}^{n} p_{\boldsymbol{Z}_{i} \mid \mathbf{G}^{n}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}^{n}\right)}\right)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) \\
& =\log \left(\frac{\prod_{i=1}^{n} p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{\prod_{i=1}^{n} p_{\boldsymbol{Z}_{i} \mid \mathbf{G}^{n}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}^{n}\right)}\right) \\
& \stackrel{(b)}{=} \log \left(\frac{\prod_{i=1}^{n} p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{\prod_{i=1}^{n} p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}\right)}\right) \\
& =\log \left(\prod_{i=1}^{n} \frac{p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}\right)}\right) \\
& =\sum_{i=1}^{n} \log \left(\frac{p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}\right)}\right) \\
& =\sum_{i=1}^{n} \log \left(\frac{p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right) p_{\mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}\right) p_{\mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}\right) \\
& \stackrel{(c)}{=} \sum_{i=1}^{n} \log \left(\frac{p_{\boldsymbol{Z}_{i}, \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i}, \mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{p_{\boldsymbol{Z}_{i}, \mathbf{G}_{i}}\left(\boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) p_{\boldsymbol{T}_{i}}\left(\boldsymbol{T}_{i}\right)}\right) \\
& =\sum_{i=1}^{n} i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right),
\end{aligned}
$$

where $(b)$ follows because conditioned on $\mathbf{G}_{i}, \boldsymbol{Z}_{i}$ is independent of $\mathbf{G}_{1}, \ldots, \mathbf{G}_{i-1}, \mathbf{G}_{i+1}, \ldots, \mathbf{G}_{n}$ since $\left(\boldsymbol{T}_{i}, \boldsymbol{\xi}_{i}\right)$ is independent of $\mathbf{G}_{1}, \ldots, \mathbf{G}_{i-1}, \mathbf{G}_{i+1}, \ldots, \mathbf{G}_{n}$ and $(c)$ follows because $\boldsymbol{T}_{i}$ and $\mathbf{G}_{i}$ are independent for $i=1 \ldots n$.

Now, recall that we chose the inputs $\boldsymbol{T}^{n}$ of $W_{\mathbf{G}^{n}}$ to be i.i.d such that $\boldsymbol{T}_{i} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{T}}, \tilde{\mathbf{Q}}\right), i=1 \ldots n$. We have using Lemma 7

$$
\begin{aligned}
\mathbb{E}\left[\frac{1}{n} i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)\right] & =\frac{1}{n} \mathbb{E}\left[\sum_{i=1}^{n} i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)\right] \\
& =\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)\right] \\
& =\frac{1}{n} \sum_{i=1}^{n} I\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) \\
& =\frac{1}{n} \sum_{i=1}^{n}\left(I\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i} \mid \mathbf{G}_{i}\right)+I\left(\boldsymbol{T}_{i}, \mathbf{G}_{i}\right)\right) \\
& =\frac{1}{n} \sum_{i=1}^{n} I\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i} \mid \mathbf{G}_{i}\right) \\
& \stackrel{(a)}{=} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)\right] \\
& \stackrel{(b)}{=} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G} \tilde{\mathbf{Q}} \mathbf{G}^{H}\right)\right] \\
& =\phi(\tilde{\mathbf{Q}})
\end{aligned}
$$

where $(a)$ follows because $\boldsymbol{\xi}_{i} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R}}, \sigma^{2} \mathbf{I}_{N_{R}}\right), i=1, \ldots, n$ and because all the $\boldsymbol{T}_{i}^{\prime} s$ are i.i.d. such that $\boldsymbol{T}_{i} \sim$ $\mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{T}}, \tilde{\mathbf{Q}}\right), i=1 \ldots n$. and $(b)$ follows because from Lemma 2, we know that vec $\left(\mathbf{G}_{i}\right) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right), i=1 \ldots n$ and because $\operatorname{vec}(\mathbf{G}) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$. It follows that

$$
\begin{align*}
& \mathbb{P}\left[\frac{1}{n} i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) \leq \phi(\tilde{\mathbf{Q}})-\frac{\delta}{2}\right] \\
& =\mathbb{P}\left[\frac{1}{n} i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) \leq \mathbb{E}\left[\frac{1}{n} i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)\right]-\frac{\delta}{2}\right] \\
& \leq \mathbb{P}\left[\left|\frac{1}{n} i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)-\mathbb{E}\left[\frac{1}{n} i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)\right]\right| \geq \frac{\delta}{2}\right] \\
& \stackrel{(a)}{\leq} \frac{4 \operatorname{var}\left(\frac{i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)}{n}\right)}{\delta^{2}} \\
& \stackrel{(b)}{\leq} \frac{4 \kappa(n)}{\delta^{2}} \tag{10}
\end{align*}
$$

where (a) follows from the Chebyshev's inequality and (b) follows because $\operatorname{var}\left(\frac{i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)}{n}\right) \leq \kappa(n)$ for some $\kappa(n)>0$ with $\lim _{n \rightarrow \infty} \kappa(n)=0$ (from the auxiliary result of Lemma 4).

From (7), (9) and (10), we obtain

$$
e_{\max }\left(\Gamma_{n}\right) \leq 4 \frac{\kappa(n)}{\delta^{2}}+2^{-n \hat{\beta}}+2^{-n \frac{\delta}{4}}
$$

where $\lim _{n \rightarrow \infty} 4 \frac{\kappa(n)}{\delta^{2}}+2^{-n \hat{\beta}}+2^{-n \frac{\delta}{4}}=0$. Therefore, for sufficiently large $n$, it holds that $e_{\max }\left(\Gamma_{n}\right) \leq \theta$. This completes the direct proof of Theorem 1.

## B. Converse Proof

Let $R$ be any achievable rate for the channel $W_{\mathbf{G}^{n}}$ in (1). So, for every $\theta, \delta>0$, there exists a code sequence $\left(\Gamma_{n}\right)_{n=1}^{\infty}$ such that

$$
\frac{\log \left\|\Gamma_{n}\right\|}{n} \geq R-\delta
$$

and

$$
\begin{equation*}
e_{\max }\left(\Gamma_{n}\right)=\max _{\ell \in\left\{1 \ldots\left\|\Gamma_{n}\right\|\right\}} \mathbb{E}\left[W_{\mathbf{G}^{n}}\left(\mathcal{D}_{\ell}^{\left(\mathbf{G}^{n}\right)_{c}} \mid \boldsymbol{t}_{\ell}\right)\right] \leq \theta \tag{11}
\end{equation*}
$$

for sufficiently large $n$.
Notice that from (11), it follows that the average error probability is also bounded from above by $\theta$. The uniformly-distributed message $M$ is mapped to the random input sequence $\boldsymbol{T}^{n}=\left(\boldsymbol{T}_{1}, \ldots, \boldsymbol{T}_{n}\right)$ of the channel in (1), where the covariance matrix of each input $\boldsymbol{T}_{i}$ is denoted by $\mathbf{Q}_{i}$. Let $\left(\boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)$ the corresponding outputs, where $\boldsymbol{Z}^{n}=\left(\boldsymbol{Z}_{1}, \ldots, \boldsymbol{Z}_{n}\right)$. We define $\mathbf{Q}^{\star}$ such that $\mathbf{Q}^{\star}=\frac{1}{n} \sum_{i=1}^{n} \mathbf{Q}_{i}$. We model the random decoded message by $\hat{M}$. The set of messages is denoted by $\mathcal{M}$.

## Lemma 8.

$$
\operatorname{tr}\left(\mathbf{Q}^{\star}\right) \leq P
$$

Proof. From (3), it holds that

$$
\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{T}_{i}^{H} \boldsymbol{T}_{i} \leq P, \quad \text { almost surely. }
$$

This implies that

$$
\begin{aligned}
\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{T}_{i}^{H} \boldsymbol{T}_{i}\right] & =\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\boldsymbol{T}_{i}^{H} \boldsymbol{T}_{i}\right] \\
& \leq P
\end{aligned}
$$

This yields

$$
\begin{aligned}
\operatorname{tr}\left[\mathbf{Q}^{\star}\right] & =\operatorname{tr}\left[\frac{1}{n} \sum_{i=1}^{n} \mathbf{Q}_{i}\right] \\
& =\frac{1}{n} \sum_{i=1}^{n} \operatorname{tr}\left[\mathbf{Q}_{i}\right] \\
& \leq \frac{1}{n} \sum_{i=1}^{n} \operatorname{tr}\left(\mathbb{E}\left[\boldsymbol{T}_{i} \boldsymbol{T}_{i}^{H}\right]\right) \\
& =\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\operatorname{tr}\left(\boldsymbol{T}_{i} \boldsymbol{T}_{i}^{H}\right)\right] \\
& =\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\operatorname{tr}\left(\boldsymbol{T}_{i}^{H} \boldsymbol{T}_{i}\right)\right] \\
& =\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\boldsymbol{T}_{i}^{H} \boldsymbol{T}_{i}\right] \\
& \leq P,
\end{aligned}
$$

where we used $r=\operatorname{tr}(r)$ for scalar $r, \operatorname{tr}(\mathbf{A B})=\operatorname{tr}(\mathbf{B A})$ and the linearity of the expectation and of the trace operators.
By using $\Gamma_{n}$ as a transmission-code for the channel $W_{\mathbf{G}^{n}}$, it follows using the fact that $M$ and $\mathbf{G}^{n}$ are independent that

$$
\begin{aligned}
\mathbb{P}[\hat{M} \neq M] & =\mathbb{E}\left[\mathbb{P}\left[M \neq \hat{M} \mid \mathbf{G}^{n}\right]\right] \\
& =\mathbb{E}\left[\sum_{\ell=1}^{|\mathcal{M}|} \mathbb{P}[M=\ell] \mathbb{P}\left[\hat{M} \neq \ell \mid M=\ell, \mathbf{G}^{n}\right]\right] \\
& =\sum_{\ell=1}^{|\mathcal{M}|} \mathbb{P}[M=\ell] \mathbb{E}\left[\mathbb{P}\left[\hat{M} \neq \ell \mid M=\ell, \mathbf{G}^{n}\right]\right] \\
& =\sum_{\ell=1}^{|\mathcal{M}|} \mathbb{P}[M=\ell] \mathbb{E}\left[W_{\mathbf{G}^{n}}\left(\mathcal{D}_{\ell}^{\left(\mathbf{G}^{n}\right)_{c}} \mid \boldsymbol{t}_{\ell}\right)\right] \\
& \leq e_{\max }\left(\Gamma_{n}\right) \\
& \leq \theta
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
H(M) & =\log |\mathcal{M}| \\
& =\log \left\|\Gamma_{n}\right\| \\
& \geq n(R-\delta) .
\end{aligned}
$$

By applying Fano's inequality, we obtain

$$
\begin{aligned}
H(M \mid \hat{M}) & \leq 1+\mathbb{P}[M \neq \hat{M}] \log |\mathcal{M}| \\
& \leq 1+\theta \log |\mathcal{M}| \\
& =1+\theta H(M) .
\end{aligned}
$$

Now, on the one hand, it holds that

$$
\begin{aligned}
I(M ; \hat{M}) & =H(M)-H(M \mid \hat{M}) \\
& \geq(1-\theta) H(M)-1,
\end{aligned}
$$

which yields

$$
H(M) \leq \frac{1+I(M ; \hat{M})}{1-\theta}
$$

On the other hand

$$
\begin{aligned}
& \frac{1}{n} I(M ; \hat{M}) \\
& \stackrel{(a)}{\leq} \frac{1}{n} I\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right) \\
& =\frac{1}{n} I\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n} \mid \mathbf{G}^{n}\right)+\frac{1}{n} I\left(\boldsymbol{T}^{n}, \mathbf{G}^{n}\right) \\
& \stackrel{(b)}{=} \frac{1}{n} I\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n} \mid \mathbf{G}^{n}\right) \\
& \stackrel{(c)}{=} \frac{1}{n} \sum_{i=1}^{n} I\left(\boldsymbol{Z}_{i} ; \boldsymbol{T}^{n} \mid \mathbf{G}^{n}, \boldsymbol{Z}^{i-1}\right) \\
& =\frac{1}{n} \sum_{i=1}^{n} h\left(\boldsymbol{Z}_{i} \mid \mathbf{G}^{n}, \boldsymbol{Z}^{i-1}\right)-h\left(\boldsymbol{Z}_{i} \mid \mathbf{G}^{n}, \boldsymbol{T}^{n}, \boldsymbol{Z}^{i-1}\right) \\
& \stackrel{(d)}{=} \frac{1}{n} \sum_{i=1}^{n} h\left(\boldsymbol{Z}_{i} \mid \mathbf{G}^{n}, \boldsymbol{Z}^{i-1}\right)-h\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right) \\
& \stackrel{(e)}{\leq} \frac{1}{n} \sum_{i=1}^{n} h\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}\right)-h\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right) \\
& =\frac{1}{n} \sum_{i=1}^{n} I\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i} \mid \mathbf{G}_{i}\right) \\
& \stackrel{(f)}{\leq} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \mathbf{Q}_{i} \mathbf{G}_{i}^{H}\right)\right] \\
& \stackrel{(\underline{g})}{\underline{n}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G} \mathbf{Q}_{i} \mathbf{G}^{H}\right)\right] \\
& =\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G} \mathbf{Q}_{i} \mathbf{G}^{H}\right)\right] \\
& \stackrel{(h)}{\leq} \mathbb{E}\left[\log \operatorname{det}\left(\frac{1}{n} \sum_{i=1}^{n}\left[\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G} \mathbf{Q}_{i} \mathbf{G}^{H}\right]\right)\right] \\
& =\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}\left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{Q}_{i}\right) \mathbf{G}^{H}\right)\right] \\
& =\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G} \mathbf{Q}^{\star} \mathbf{G}^{H}\right)\right] \\
& \stackrel{(i)}{\leq} \max _{\mathbf{Q} \in \mathcal{Q}_{\left(P, N_{T}\right)}} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G Q G}^{H}\right)\right] \text {, }
\end{aligned}
$$

where (a) follows from the Data Processing Inequality because $M \ominus \boldsymbol{T}^{n} \ominus \mathbf{G}^{n}, \boldsymbol{Z}^{n} \ominus \hat{M}$ forms a Markov chain, (b) follows because $\mathbf{G}^{n}$ and $\boldsymbol{T}^{n}$ are independent, (c) follows from the chain rule for mutual information, $(d)$ follows because

$$
\mathbf{G}_{1} \boldsymbol{T}_{1} \ldots \mathbf{G}_{i-1} \boldsymbol{T}_{i-1} \mathbf{G}_{i+1} \boldsymbol{T}_{i+1} \ldots \mathbf{G}_{n} \boldsymbol{T}_{n} \boldsymbol{Z}^{i-1} \ominus \mathbf{G}_{i} \boldsymbol{T}_{i} \ominus \boldsymbol{Z}_{i}
$$

forms a Markov chain, ( $e$ ) follows because conditioning does not increase entropy, $(f)$ follows because $\boldsymbol{\xi}_{i} \sim$ $\mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R}}, \sigma^{2} \mathbf{I}_{N_{R}}\right), i=1 \ldots n,(g)$ follows because the $\mathbf{G}_{i} s$ are identically distributed from Lemma 2 where $\mathbf{G}$ is a random matrix that has the same distribution as each of the $\mathbf{G}_{i}$ and $(h)$ follows from Jensen's Inequality since the function log o det is concave on the set of Hermitian positive semidefinite matrices and since $\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G} \mathbf{Q}_{i} \mathbf{G}^{H}$ is Hermitian positive semidefinite for $i=1 \ldots n$, ( $i$ ) follows because $\mathbf{Q}^{\star}=\frac{1}{n} \sum_{i=1}^{n} \mathbf{Q}_{i} \in \mathcal{Q}_{\left(P, N_{T}\right)}$ from Lemma 8.

As a result, we have

$$
n(R-\delta) \leq \frac{n \max _{\mathbf{Q} \in \mathcal{Q}_{\left(P, N_{T}\right)}} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G Q G}{ }^{H}\right)\right]+1}{1-\theta}
$$

This implies that

$$
\begin{equation*}
R \leq \frac{\max _{\mathbf{Q} \in \mathcal{Q}_{\left(P, N_{T}\right)}} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G Q G}{ }^{H}\right)\right]+\frac{1}{n}}{1-\theta}+\delta \tag{12}
\end{equation*}
$$

In particular, we can choose $\delta, \theta>0$ to be arbitrarily small such that the right-hand side of (12) is equal to $\max _{\mathbf{Q} \in \mathcal{Q}_{\left(P, N_{T}\right)}} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G Q G}^{H}\right)\right]+\delta^{\prime}$ for $n \rightarrow \infty$, with $\delta^{\prime}$ being an arbitrarily small positive constant. This completes the converse proof of Theorem 1 .

## IV. Proof of Lemma 4

Let $\boldsymbol{T}^{n}=\left(\boldsymbol{T}_{1}, \ldots, \boldsymbol{T}_{n}\right)$ be an $n$-length input sequence of the channel $W_{\mathbf{G}^{n}}$ such that the $\boldsymbol{T}_{i}^{\prime} s$ are i.i.d., where $\boldsymbol{T}_{i} \sim$ $\mathcal{N}\left(\mathbf{0}_{N_{T}}, \tilde{\mathbf{Q}}\right), i=1 \ldots n$ and where $\tilde{\mathbf{Q}} \in \mathcal{Q}_{\left(P, N_{T}\right)}$. Let $\boldsymbol{Z}^{n}$ be the corresponding output sequence, where $\boldsymbol{Z}^{n}=\left(\boldsymbol{Z}_{1}, \ldots, \boldsymbol{Z}_{n}\right)$. By Lemma 7, it holds that

$$
\begin{equation*}
i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)=\sum_{i=1}^{n} i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) \tag{13}
\end{equation*}
$$

We have

$$
\begin{equation*}
\operatorname{var}\left(\frac{i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)}{n}\right)=\frac{1}{n^{2}} \mathbb{E}\left[i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)^{2}\right]-\frac{1}{n^{2}} \mathbb{E}\left[i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)\right]^{2} \tag{14}
\end{equation*}
$$

Let $\tilde{\mathbf{G}}$ be any random matrix with i.i.d. entries, independent of $\mathbf{G}_{1}$ and $\mathbf{W}_{i}, i=2, \ldots n$ such that $\operatorname{vec}(\tilde{\mathbf{G}}) \sim$ $\mathcal{N}_{\widetilde{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$. By Lemma 2, it follows that $\tilde{\mathbf{G}}$ has the same distribution as $\mathbf{G}_{i}, i=1, \ldots n$. Furthermore, since $\tilde{\mathbf{G}}$ is independent of $\mathbf{G}_{1}$ and $\mathbf{W}_{i}, i=2, \ldots n$, it is also independent of all the $\mathbf{G}_{i} s$. Now

$$
\begin{align*}
& \frac{1}{n^{2}} \mathbb{E}\left[i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)^{2}\right] \\
& =\frac{1}{n^{2}} \mathbb{E}\left[\left(\sum_{i=1}^{n} i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)\right)^{2}\right] \\
& =\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) i\left(\boldsymbol{T}_{k} ; \boldsymbol{Z}_{k}, \mathbf{G}_{k}\right)\right]+\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right] \\
& =\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E}\left[\mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) i\left(\boldsymbol{T}_{k} ; \boldsymbol{Z}_{k}, \mathbf{G}_{k}\right) \mid \mathbf{G}_{i}, \mathbf{G}_{k}\right]\right]+\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right] \\
& \stackrel{(a)}{=} \frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E}\left[\mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) \mid \mathbf{G}_{i}, \mathbf{G}_{k}\right] \mathbb{E}\left[i\left(\boldsymbol{T}_{k} ; \boldsymbol{Z}_{k}, \mathbf{G}_{k}\right) \mid \mathbf{G}_{i}, \mathbf{G}_{k}\right]\right]+\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right] \\
& \stackrel{(b)}{=} \frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E}\left[\mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) \mid \mathbf{G}_{i}\right] \mathbb{E}\left[i\left(\boldsymbol{T}_{k} ; \boldsymbol{Z}_{k}, \mathbf{G}_{k}\right) \mid \mathbf{G}_{k}\right]\right]+\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right] \\
& \stackrel{(c)}{=} \frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right) \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{k} \tilde{\mathbf{Q}} \mathbf{G}_{k}^{H}\right)\right]+\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right], \tag{15}
\end{align*}
$$

where $(a)$ follows because for independent inputs and conditioned on $\left(\mathbf{G}_{i}, \mathbf{G}_{j}\right), i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)$ and $i\left(\boldsymbol{T}_{j} ; \boldsymbol{Z}_{j}, \mathbf{G}_{j}\right)$ are independent, ( $b$ ) follows because for independent inputs and conditioned on $\mathbf{G}_{i}, i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)$ and $\mathbf{G}_{k}$ are independent, and because for independent inputs and conditioned on $\mathbf{G}_{k}, i\left(\boldsymbol{T}_{k} ; \boldsymbol{Z}_{k}, \mathbf{G}_{k}\right)$ and $\mathbf{G}_{i}$ are independent, and (c) follows because $\boldsymbol{\xi}_{i} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R}}, \sigma^{2} \mathbf{I}_{N_{R}}\right), i=1, \ldots, n$ and because all the $\boldsymbol{T}_{i} s$ are i.i.d. such that $\boldsymbol{T}_{i} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{T}}, \tilde{\mathbf{Q}}\right), i=1 \ldots n$.

It holds using (13) that

$$
\begin{align*}
\frac{1}{n^{2}} \mathbb{E}\left[i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)\right]^{2} & =\frac{1}{n^{2}} \mathbb{E}\left[\sum_{i=1}^{n} i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)\right]^{2} \\
& =\frac{1}{n^{2}}\left(\sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)\right]\right)^{2} \\
& \geq \frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)\right] \mathbb{E}\left[i\left(\boldsymbol{T}_{k} ; \boldsymbol{Z}_{k}, \mathbf{G}_{k}\right)\right] \\
& =\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E}\left[\mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) \mid \mathbf{G}_{i}\right]\right] \mathbb{E}\left[\mathbb{E}\left[i\left(\boldsymbol{T}_{k} ; \boldsymbol{Z}_{k}, \mathbf{G}_{k}\right) \mid \mathbf{G}_{k}\right]\right] \\
& =\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)\right] \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{k} \tilde{\mathbf{Q}} \mathbf{G}_{k}^{H}\right)\right] \\
& =\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right)\right]^{2} . \tag{16}
\end{align*}
$$

It follows from (14), (15) and (16) that

$$
\begin{align*}
& \operatorname{var}\left(\frac{i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)}{n}\right) \\
& \leq \frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right) \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{k} \tilde{\mathbf{Q}} \mathbf{G}_{k}^{H}\right)\right] \\
& \quad+\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right]-\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right)\right]^{2} \\
& =\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n}\left(\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right) \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{k} \tilde{\mathbf{Q}} \mathbf{G}_{k}^{H}\right)\right]-\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right)\right]^{2}\right) \\
& \quad+\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right] . \tag{17}
\end{align*}
$$

By defining for any $i, k \in\{1, \ldots n\}$ with $i \neq k$,

$$
m(i, k)=\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right) \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{k} \tilde{\mathbf{Q}} \mathbf{G}_{k}^{H}\right)\right]-\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right)\right]^{2},
$$

we obtain using (17)

$$
\begin{aligned}
& \operatorname{var}\left(\frac{i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)}{n}\right) \\
& \leq \frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} m(i, k)+\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right]
\end{aligned}
$$

$$
\begin{equation*}
=\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} m(i, k)+\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k)+\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right] . \tag{18}
\end{equation*}
$$

Now, the goal is to find a suitable upper-bound for each term in (18).
A. Upper-bound for $\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} m(i, k)+\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k)$

We are going to show that

$$
\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} m(i, k)+\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k) \leq \frac{2 c^{\prime}}{n(1-\sqrt{\alpha})}
$$

for some $c^{\prime}>0$. Let us first introduce and prove the following Lemma
Lemma 9. Let $i_{1}, i_{2} \in\{1, \ldots n\}$. Assume without loss of generality that $i_{1}<i_{2}$, then

$$
\begin{aligned}
& \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H}\right) \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right)\right] \\
& \leq \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right)\right]^{2}+c^{\prime} \sqrt{\alpha}^{i_{2}-i_{1}}
\end{aligned}
$$

for some $c^{\prime}>0$, where $\tilde{\mathbf{G}}$ is a random matrix with i.i.d. entries, independent of $\mathbf{G}_{1}$ and $\mathbf{W}_{i}, i=2, \ldots n$, such that $\operatorname{vec}(\tilde{\mathbf{G}}) \sim$ $\mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$.

Proof. By Lemma 3, we know that

$$
\mathbf{G}_{i_{2}}=\sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}}+\sqrt{1-\alpha} \sum_{j=i_{1}+1}^{i_{2}} \sqrt{\alpha}^{i_{2}-j} \mathbf{W}_{j}
$$

By defining

$$
\mathbf{S}=\sqrt{1-\alpha} \sum_{j=i_{1}+1}^{i_{2}} \sqrt{\alpha}^{i_{2}-j} \mathbf{W}_{j}
$$

it follows that

$$
\begin{equation*}
\mathbf{G}_{i_{2}}=\sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}}+\mathbf{S} \tag{19}
\end{equation*}
$$

Define

$$
\tilde{\mathbf{W}}=\mathbf{S}+\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}}
$$

with $\tilde{\mathbf{G}}$ being a random matrix with i.i.d. entries, independent of $\mathbf{G}_{1}$ and $\mathbf{W}_{i}, i=2, \ldots n$ such that $\operatorname{vec}(\tilde{\mathbf{G}}) \sim$ $\mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$.

Since $\mathbf{W}_{i}, i=i_{1}+1, \ldots, i_{2}$, have i.i.d entries, it follows that $\tilde{\mathbf{W}}$ has i.i.d. entries. Notice also that $\tilde{\mathbf{W}}$ is independent of $\mathbf{G}_{i_{1}}$, since $\mathbf{G}_{i_{1}}$ is independent of $(\mathbf{S}, \tilde{\mathbf{G}})$. Analogously to the proof of Lemma 2, one can show that $\operatorname{vec}(\tilde{\mathbf{W}}) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$.

The proof of Lemma 9 is divided in three parts:

1) We will prove first that

$$
\begin{aligned}
& \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H}\right) \\
& \leq \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right)+\frac{N_{R}}{\ln (2) \sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right)
\end{aligned}
$$

2) We will prove second that

$$
\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right) \leq \frac{P N_{R}}{\ln (2) \sigma^{2}}\left\|\mathbf{G}_{i_{1}}\right\|^{2}
$$

3) This will allow us to show that

$$
\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H}\right) \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right)\right] \leq \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right)\right]^{2}+c^{\prime} \sqrt{\alpha}^{i_{2}-i_{1}}
$$

for some $c^{\prime}>0$.

1) Upper-bound for $\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H}\right)$ : From (19), we have

$$
\begin{align*}
& \frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H} \\
& =\frac{1}{\sigma^{2}}\left[\sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}}+\mathbf{S}\right] \tilde{\mathbf{Q}}\left[\sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}}^{H}+\mathbf{S}^{H}\right] \\
& =\frac{1}{\sigma^{2}} \alpha^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{S}^{H}+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{S} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\frac{1}{\sigma^{2}} \mathbf{S} \tilde{\mathbf{Q}} \mathbf{S}^{H} \\
& =\frac{1}{\sigma^{2}}\left[\alpha^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\mathbf{S} \tilde{\mathbf{Q}} \mathbf{S}^{H}\right]+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{S}^{H}+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{S} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H} . \tag{20}
\end{align*}
$$

We will prove first that

$$
\begin{align*}
& \frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{S}^{H}+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{S} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H} \\
& \preceq \frac{2 P}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\| \mathbf{I}_{N_{R}} \tag{21}
\end{align*}
$$

From Lemma 10 in the Appendix, we know that for any Hermitian matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$, it holds that $\mathbf{A} \preceq\|\mathbf{A}\| \mathbf{I}_{n}$. Notice now that the matrix

$$
\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{S}^{H}+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{S} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}
$$

is a Hermitian matrix since it is equal to its Hermitian transpose. It follows using Lemma 10 in the Appendix that

$$
\begin{align*}
& \frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{S}^{H}+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{S} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H} \\
& \preceq\left\|\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{S}^{H}+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{S} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right\| \mathbf{I}_{N_{R}} \\
& \preceq\left(\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left\|\mathbf{G}_{i_{1}}\right\|\|\tilde{\mathbf{Q}}\|\left\|\mathbf{S}^{H}\right\|+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\|\mathbf{S}\|\|\tilde{\mathbf{Q}}\|\left\|\mathbf{G}_{i_{1}}^{H}\right\|\right) \mathbf{I}_{N_{R}} \\
& =\frac{2}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left\|\mathbf{G}_{i_{1}}\right\|\|\tilde{\mathbf{Q}}\|\|\mathbf{S}\| \mathbf{I}_{N_{R}} \\
& \preceq \frac{2 P}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\| \mathbf{I}_{N_{R}} . \tag{22}
\end{align*}
$$

This proves (21).
Next, we will prove that

$$
\begin{align*}
& \frac{1}{\sigma^{2}}\left[\alpha^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\mathbf{S} \tilde{\mathbf{Q}} \mathbf{S}^{H}\right] \\
& \preceq \frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\frac{P}{\sigma^{2}} \alpha^{i_{2}-i_{1}}\left(\left\|\mathbf{G}_{i_{1}}\right\|^{2}+\|\tilde{\mathbf{G}}\|^{2}\right) \mathbf{I}_{N_{R}}+\frac{2}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\| \mathbf{I}_{N_{R}} \tag{23}
\end{align*}
$$

It follows using the fact that $\tilde{\mathbf{W}}=\mathbf{S}+\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}}$ that

$$
\begin{aligned}
& \mathbf{S} \mathbf{Q}^{H} \\
& =\left(\mathbf{S}+\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}}-\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}}\right) \tilde{\mathbf{Q}}\left(\mathbf{S}^{H}+\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}}^{H}-\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}}^{H}\right) \\
& =\left(\tilde{\mathbf{W}}-\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}}\right) \tilde{\mathbf{Q}}\left(\tilde{\mathbf{W}}^{H}-\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}}^{H}\right) \\
& =\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}-\sqrt{\alpha}{ }^{i_{2}-i_{1}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}-\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\alpha^{i_{2}-i_{1}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H} .
\end{aligned}
$$

This yields

$$
\begin{align*}
& \frac{1}{\sigma^{2}}\left[\alpha^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\mathbf{S} \tilde{\mathbf{Q}} \mathbf{S}^{H}\right] \\
& =\frac{1}{\sigma^{2}}\left[\alpha^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}-\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}-\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\alpha^{i_{2}-i_{1}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right] \\
& =\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}  \tag{24}\\
& H \\
& H \\
& \sigma^{2}
\end{align*} \alpha^{i_{2}-i_{1}}\left[\mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right]-\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}-\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}{ }^{H} .
$$

Now notice that

$$
\frac{1}{\sigma^{2}} \alpha^{i_{2}-i_{1}}\left[\mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right]
$$

is a Hermitian matrix. This implies using Lemma 10 that

$$
\begin{align*}
& \frac{1}{\sigma^{2}} \alpha^{i_{2}-i_{1}}\left[\mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right] \\
& \preceq \frac{1}{\sigma^{2}} \alpha^{i_{2}-i_{1}}\left\|\mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\| \mathbf{I}_{N_{R}} \\
& \preceq \frac{1}{\sigma^{2}} \alpha^{i_{2}-i_{1}}\|\tilde{\mathbf{Q}}\|\left(\left\|\mathbf{G}_{i_{1}}\right\|^{2}+\|\tilde{\mathbf{G}}\|^{2}\right) \mathbf{I}_{N_{R}} \\
& \preceq \frac{P}{\sigma^{2}} \alpha^{i_{2}-i_{1}}\left(\left\|\mathbf{G}_{i_{1}}\right\|^{2}+\|\tilde{\mathbf{G}}\|^{2}\right) \mathbf{I}_{N_{R}} . \tag{25}
\end{align*}
$$

Notice also that $-\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}-\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}$ is a Hermitian matrix. It follows using Lemma 10 that

$$
\begin{align*}
& -\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}-\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H} \\
& \preceq\left\|-\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}-\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right\| \mathbf{I}_{N_{R}} \\
& \preceq \frac{2}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\| \mathbf{I}_{N_{R}} . \tag{26}
\end{align*}
$$

As a result, we have using (25) and (26)

$$
\begin{align*}
& \frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\frac{1}{\sigma^{2}} \alpha^{i_{2}-i_{1}}\left[\mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right]-\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}-\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H} \\
& \preceq \frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\frac{P}{\sigma^{2}} \alpha^{i_{2}-i_{1}}\left(\left\|\mathbf{G}_{i_{1}}\right\|^{2}+\|\tilde{\mathbf{G}}\|^{2}\right) \mathbf{I}_{N_{R}}+\frac{2}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\| \mathbf{I}_{N_{R}} . \tag{27}
\end{align*}
$$

This proves (23). Thus, it follows from (24) and (27) that

$$
\begin{align*}
& \frac{1}{\sigma^{2}}\left[\alpha^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\mathbf{S} \tilde{\mathbf{Q}} \mathbf{S}^{H}\right] \\
& \preceq \frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\frac{P}{\sigma^{2}} \alpha^{i_{2}-i_{1}}\left(\left\|\mathbf{G}_{i_{1}}\right\|^{2}+\|\tilde{\mathbf{G}}\|^{2}\right) \mathbf{I}_{N_{R}}+\frac{2}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\| \mathbf{I}_{N_{R}} \tag{28}
\end{align*}
$$

We deduce using (22) and (28) that

$$
\begin{align*}
& \frac{1}{\sigma^{2}}\left[\alpha^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}+\mathbf{S} \tilde{\mathbf{Q}} \tilde{\mathbf{S}}\right]+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{S}^{H}+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbf{S} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H} \\
& \preceq \frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\frac{P}{\sigma^{2}} \alpha^{i_{2}-i_{1}}\left(\left\|\mathbf{G}_{i_{1}}\right\|^{2}+\|\tilde{\mathbf{G}}\|^{2}\right) \mathbf{I}_{N_{R}}+\frac{2}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\| \mathbf{I}_{N_{R}}+\frac{2 P}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\| \mathbf{I}_{N_{R}} \\
& =\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\frac{P}{\sigma^{2}} \alpha^{i_{2}-i_{1}}\left(\left\|\mathbf{G}_{i_{1}}\right\|^{2}+\|\tilde{\mathbf{G}}\|^{2}\right) \mathbf{I}_{N_{R}}+\frac{2}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right) \mathbf{I}_{N_{R}} \\
& \stackrel{(a)}{\preceq} \frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\frac{P}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(\left\|\mathbf{G}_{i_{1}}\right\|^{2}+\|\tilde{\mathbf{G}}\|^{2}\right) \mathbf{I}_{N_{R}}+\frac{2}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}}^{H}\right\|+P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right) \mathbf{I}_{N_{R}} \\
& =\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right) \mathbf{I}_{N_{R}}, \tag{29}
\end{align*}
$$

where ( $a$ ) follows because $\alpha<\sqrt{\alpha}$ for $0<\alpha<1$.
Therefore, it follows from (20) and (29) that

$$
\begin{aligned}
& \frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H} \\
& \preceq \frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right) \mathbf{I}_{N_{R}}
\end{aligned}
$$

This yields

$$
\begin{align*}
& \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H}\right) \\
& \leq \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}+\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right) \mathbf{I}_{N_{R}}\right) \tag{30}
\end{align*}
$$

Now by Lemma 11 in the Appendix, we know that for any positive-definite Hermitian matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ with smallest eigenvalue $\lambda_{\min }(\mathbf{A})$ and for any positive semi-definite Hermitian matrix $\mathbf{B} \in \mathbb{C}^{n \times n}$, the following is satisfied:

$$
\log \operatorname{det}(\mathbf{A}+\mathbf{B}) \leq \log \operatorname{det}(\mathbf{A})+\log \operatorname{det}\left(\mathbf{I}_{n}+\frac{1}{\lambda_{\min }(\mathbf{A})} \mathbf{B}\right)
$$

By applying Lemma 11 in the Appendix for

$$
\mathbf{A}=\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}
$$

and for

$$
\mathbf{B}=\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right) \mathbf{I}_{N_{R}}
$$

it follows from (30) that

$$
\begin{aligned}
& \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H}\right) \\
& \leq \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right) \\
& +\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right)}{\lambda_{\min }\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right)} \mathbf{I}_{N_{R}}\right) \\
& \stackrel{(a)}{\leq} \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right) \\
& +\frac{1}{\ln (2)} \operatorname{tr}\left[\frac{\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right)}{\lambda_{\min }\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right)} \mathbf{I}_{N_{R}}\right] \\
& \stackrel{(b)}{\leq} \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right) \\
& +\frac{1}{\ln (2)} \operatorname{tr}\left[\frac{1}{\sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right) \mathbf{I}_{N_{R}}\right] \\
& \stackrel{(c)}{=} \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right) \\
& +\frac{N_{R}}{\ln (2) \sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right),
\end{aligned}
$$

where $(a)$ follows because $\ln \operatorname{det}\left(\mathbf{I}_{n}+\mathbf{A}\right) \leq \operatorname{tr}(\mathbf{A})$ for positive semi-definite $\mathbf{A}$, (b) follows because $\lambda_{\min }\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right) \geq 1$ and $(c)$ follows because $\operatorname{tr}\left(c \mathbf{I}_{N_{R}}\right)=c N_{R}$ for any constant $c$.

To conclude, we have proved that

$$
\begin{align*}
& \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H}\right) \\
& \leq \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right)+\frac{N_{R}}{\ln (2) \sigma^{2}} \sqrt{\alpha}^{i_{2}-i_{1}}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right) \tag{31}
\end{align*}
$$

2) Upper-bound for $\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right)$ : By Lemma 10 in the Appendix, we have

$$
\begin{aligned}
\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right) & \leq \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}}\left\|\mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right\| \mathbf{I}_{N_{R}}\right) \\
& \leq \frac{1}{\ln (2)} \operatorname{tr}\left[\frac{1}{\sigma^{2}}\left\|\mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right\| \mathbf{I}_{N_{R}}\right] \\
& \leq \frac{1}{\ln (2)} \operatorname{tr}\left[\frac{1}{\sigma^{2}}\left\|\mathbf{G}_{i_{1}}\right\|^{2}\|\tilde{\mathbf{Q}}\| \mathbf{I}_{N_{R}}\right]
\end{aligned}
$$

$$
\begin{align*}
& =\frac{N_{R}}{\ln (2) \sigma^{2}}\left\|\mathbf{G}_{i_{1}}\right\|^{2}\|\tilde{\mathbf{Q}}\| \\
& \leq \frac{P N_{R}}{\ln (2) \sigma^{2}}\left\|\mathbf{G}_{i_{1}}\right\|^{2} . \tag{32}
\end{align*}
$$

3) Upper bound for $\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H}\right) \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right)\right]$ : Let

$$
\Lambda\left(\mathbf{G}_{i_{1}}, \mathbf{S}, \tilde{\mathbf{G}}, \tilde{\mathbf{W}}\right)=\left\|\mathbf{G}_{i_{1}}\right\|^{2}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right)
$$

It follows using (31) and (32) that

$$
\begin{aligned}
& \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H}\right) \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right)\right] \\
& \leq \\
& \\
& \quad+\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right) \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right)\right] \\
& = \\
& \ln (2) \sigma^{2} \\
& \left.\mathbb{E}^{i_{2}-i_{1}}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right) \frac{P N_{R}}{\ln (2) \sigma^{2}}\left\|\mathbf{G}_{i_{1}}\right\|^{2}\right] \\
& \quad+\mathbb{E}\left[\frac{P N_{R}^{2}}{\ln (2)^{2} \sigma^{4}} \sqrt{\alpha}^{i_{2}-i_{1}}\left\|\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}\right\|^{2}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right)\right]
\end{aligned}
$$

$$
\stackrel{(a)}{=} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{W}}^{H}\right)\right] \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right)\right]+\frac{P N_{R}^{2}}{\ln (2)^{2} \sigma^{4}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbb{E}\left[\Lambda\left(\mathbf{G}{i_{1}}_{1}, \mathbf{S}, \tilde{\mathbf{G}}, \tilde{\mathbf{W}}\right)\right]
$$

$$
\stackrel{(b)}{=} \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right)\right]^{2}+\frac{P N_{R}^{2}}{\ln (2)^{2} \sigma^{4}} \sqrt{\alpha}^{i_{2}-i_{1}} \mathbb{E}\left[\Lambda\left(\mathbf{G}_{i_{1}}, \mathbf{S}, \tilde{\mathbf{G}}, \tilde{\mathbf{W}}\right)\right]
$$

where $(a)$ follows because $\tilde{\mathbf{W}}$ and $\mathbf{G}_{i_{1}}$ are independent, $(b)$ follows because $\tilde{\mathbf{G}}$ has the same distribution as $\tilde{\mathbf{W}}$ and $\mathbf{G}_{i_{1}}$ since $\operatorname{vec}(\tilde{\mathbf{W}}) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$ and since from Lemma 2, we know that $\operatorname{vec}\left(\mathbf{G}_{i_{1}}\right) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$.

Now, from Lemma 12 in the Appendix we know that $\mathbb{E}\left[\Lambda\left(\mathbf{G}_{i_{1}}, \mathbf{S}, \tilde{\mathbf{G}}, \tilde{\mathbf{W}}\right)\right]$ is bounded from above by some $c>0$. Therefore it follows that for $i_{1}<i_{2}$

$$
\begin{aligned}
& \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{2}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{2}}^{H}\right) \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i_{1}} \tilde{\mathbf{Q}} \mathbf{G}_{i_{1}}^{H}\right)\right] \\
& \leq \mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right)\right]^{2}+\frac{P N_{R}^{2}}{\ln (2)^{2} \sigma^{4}} c \sqrt{\alpha}^{i_{2}-i_{1}} \\
& =\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right)\right]^{2}+c^{\prime} \sqrt{\alpha}^{i_{2}-i_{1}}
\end{aligned}
$$

for some $c>0$, where $c^{\prime}=\frac{P N_{R}^{2} c}{\ln (2)^{2} \sigma^{4}}>0$. This completes the proof of Lemma 9.
Now that we proved Lemma 9, we will use that lemma to prove that

$$
\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} m(i, k)+\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k) \leq \frac{2 c^{\prime}}{n(1-\sqrt{\alpha})}
$$

We recall that for any $i, k \in\{1, \ldots n\}$ with $i \neq k$,

$$
m(i, k)=\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right) \log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{k} \tilde{\mathbf{Q}} \mathbf{G}_{k}^{H}\right)\right]-\mathbb{E}\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right)\right]^{2}
$$

If $k<i$ : Lemma 9 implies that

$$
m(i, k) \leq c^{\prime} \sqrt{\alpha}^{i-k}
$$

If $i<k$ : Lemma 9 implies that

$$
m(i, k) \leq c^{\prime} \sqrt{\alpha}^{k-i}
$$

Therefore, we have

$$
\begin{align*}
& \frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} m(i, k) \\
& \leq \frac{c^{\prime}}{n^{2}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} \sqrt{\alpha}^{i-k} \\
& \leq \frac{c^{\prime}}{n(1-\sqrt{\alpha})}, \tag{33}
\end{align*}
$$

because by Lemma 15 in the Appendix, we have for any $0<\alpha<1$

$$
\sum_{i=1}^{n} \sum_{k=1}^{i-1} \alpha^{i-k} \leq \frac{n}{1-\alpha}
$$

Furthermore, it holds that

$$
\begin{align*}
\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k) & \leq \frac{c^{\prime}}{n^{2}} \sum_{i=1}^{n} \sum_{k=i+1}^{n} \sqrt{\alpha}^{k-i} \\
& \leq \frac{c^{\prime}}{n(1-\sqrt{\alpha})} \tag{34}
\end{align*}
$$

because by Lemma 16 in the Appendix, we have for any $0<\alpha<1$

$$
\sum_{i=1}^{n} \sum_{k=i+1}^{n} \alpha^{k-i} \leq \frac{n}{1-\alpha}
$$

From (33) and (34), we deduce that

$$
\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} m(i, k)+\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k) \leq \frac{2 c^{\prime}}{n(1-\sqrt{\alpha})}
$$

B. Upper-bound for $\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right]$

We are going to prove that

$$
\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right] \leq \frac{c^{\prime \prime}}{n}
$$

for some $c^{\prime \prime}>0$. It suffices to show that $\mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right]$ is bounded from above for $i=1, \ldots, n$. Recall that

$$
\boldsymbol{Z}_{i}=\mathbf{G}_{i} \boldsymbol{T}_{i}+\boldsymbol{\xi}_{i}, \quad i=1 \ldots n
$$

and that for $i=1 \ldots n$

$$
\boldsymbol{\xi}_{i} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R}}, \sigma^{2} \mathbf{I}_{N_{R}}\right) .
$$

By Lemma 17 in the Appendix, we know that for $i=1, \ldots, n$

$$
\begin{aligned}
& i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) \\
& =\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)-\frac{1}{\ln (2) \sigma^{2}}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)^{H}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)+\frac{1}{\ln (2) \sigma^{2}} \boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G} \tilde{\mathbf{Q}} \mathbf{G}^{H}\right)^{-1} \boldsymbol{Z}_{i}
\end{aligned}
$$

We have

$$
\begin{aligned}
& \left|i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)\right| \\
& =\left|\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)-\frac{1}{\ln (2) \sigma^{2}}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)^{H}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)+\frac{1}{\ln (2) \sigma^{2}} \boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G} \tilde{\mathbf{Q}} \mathbf{G}^{H}\right)^{-1} \boldsymbol{Z}_{i}\right| \\
& \leq\left|\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)+\frac{1}{\ln (2) \sigma^{2}} \boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1} \boldsymbol{Z}_{i}\right|+\frac{1}{\ln (2) \sigma^{2}}\left|\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)^{H}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)\right| \\
& =\left|\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)+\frac{1}{\ln (2) \sigma^{2}} \boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1} \boldsymbol{Z}_{i}\right|+\frac{1}{\ln (2) \sigma^{2}}\left|\boldsymbol{\xi}_{i}^{H} \boldsymbol{\xi}_{i}\right| \\
& =\left|\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)+\frac{1}{\ln (2) \sigma^{2}} \boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1} \boldsymbol{Z}_{i}\right|+\frac{1}{\ln (2) \sigma^{2}}\left\|\boldsymbol{\xi}_{i}\right\|^{2} .
\end{aligned}
$$

Since $i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) \in \mathbb{R}$, we have

$$
\begin{aligned}
& i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2} \\
& =\left|i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)\right|^{2} \\
& \leq\left(\left\lfloor\left.\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)+\frac{1}{\ln (2) \sigma^{2}} \boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1} \boldsymbol{Z}_{i} \right\rvert\,+\frac{1}{\ln (2) \sigma^{2}}\left\|\boldsymbol{\xi}_{i}\right\|^{2}\right)^{2}\right. \\
& \leq 2\left(\left\lfloor\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)+\frac{1}{\ln (2) \sigma^{2}} \boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1} \boldsymbol{Z}_{i}\left\|^{(a)}+\frac{2}{\ln (2)^{2} \sigma^{4}}\right\| \boldsymbol{\xi}_{i} \|^{4}\right.\right. \\
& \leq 4\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)\right]^{2}+\frac{4}{\ln (2)^{2} \sigma^{4}}\left\|\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1}\right\|^{2}\left\|\boldsymbol{Z}_{i}\right\|^{4}+\frac{2}{\ln (2)^{2} \sigma^{4}}\left\|\boldsymbol{\xi}_{i}\right\|^{4} \\
& \leq 4\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)\right]^{2}+\frac{4}{\ln (2)^{2} \sigma^{4}}\left(\boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1} \boldsymbol{Z}_{i}\right)^{2}+\frac{2}{\ln (2)^{2} \sigma^{4}}\left\|\boldsymbol{\xi}_{i}\right\|^{4} \\
& \leq(c) \\
& \leq 4\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)\right]^{2}+\frac{4}{\ln (2)^{2} \sigma^{4}}\left\|\boldsymbol{Z}_{i}\right\|^{4}+\frac{2}{\ln (2)^{2} \sigma^{4}}\left\|\boldsymbol{\xi}_{i}\right\|^{4} \\
& \leq 4\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)\right]^{2}+\frac{4}{\ln (2)^{2} \sigma^{4}}\left\|\mathbf{G}_{i} \boldsymbol{T}_{i}+\boldsymbol{\xi}_{i}\right\|^{4}+\frac{2}{\ln (2)^{2} \sigma^{4}}\left\|\boldsymbol{\xi}_{i}\right\|^{4} \\
& \leq 4\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)\right]^{2}+\frac{4}{\ln (2)^{2} \sigma^{4}}\left(\left\|\mathbf{G}_{i}\right\|\left\|\boldsymbol{T}_{i}\right\|+\left\|\boldsymbol{\xi}_{i}\right\|\right)^{4}+\frac{2}{\ln (2)^{2} \sigma^{4}}\left\|\boldsymbol{\xi}_{i}\right\|^{4}
\end{aligned}
$$

$$
\begin{aligned}
& \stackrel{(d)}{\leq} 4\left[\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}}\left\|\mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right\| \mathbf{I}_{N_{R}}\right)\right]^{2}+\frac{4}{\ln (2)^{2} \sigma^{4}}\left(2\left\|\mathbf{G}_{i}\right\|^{2}\left\|\boldsymbol{T}_{i}\right\|^{2}+2\left\|\boldsymbol{\xi}_{i}\right\|^{2}\right)^{2}+\frac{2}{\ln (2)^{2} \sigma^{4}}\left\|\boldsymbol{\xi}_{i}\right\|^{4} \\
& \stackrel{(e)}{\leq} \frac{4}{\ln (2)^{2}}\left[\operatorname{tr}\left(\frac{1}{\sigma^{2}}\left\|\mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right\| \mathbf{I}_{N_{R}}\right)\right]^{2}+\frac{32}{\ln (2)^{2} \sigma^{4}}\left(\left\|\mathbf{G}_{i}\right\|^{4}\left\|\boldsymbol{T}_{i}\right\|^{4}+\left\|\boldsymbol{\xi}_{i}\right\|^{4}\right)+\frac{2}{\ln (2)^{2} \sigma^{4}}\left\|\boldsymbol{\xi}_{i}\right\|^{4} \\
& \leq \frac{4}{\ln (2)^{2} \sigma^{4}} N_{R}^{2}\left\|\mathbf{G}_{i}\right\|^{4}\|\tilde{\mathbf{Q}}\|^{2}+\frac{32}{\ln (2)^{2} \sigma^{4}}\left(\left\|\mathbf{G}_{i}\right\|^{4}\left\|\boldsymbol{T}_{i}\right\|^{4}+\left\|\boldsymbol{\xi}_{i}\right\|^{4}\right)+\frac{2}{\ln (2)^{2} \sigma^{4}}\left\|\boldsymbol{\xi}_{i}\right\|^{4} \\
& \stackrel{(f)}{\leq} \frac{4}{\ln (2)^{2} \sigma^{4}} N_{R}^{2} P^{2}\left\|\mathbf{G}_{i}\right\|^{4}+\frac{32}{\ln (2)^{2} \sigma^{4}}\left(\left\|\mathbf{G}_{i}\right\|^{4}\left\|\boldsymbol{T}_{i}\right\|^{4}+\left\|\boldsymbol{\xi}_{i}\right\|^{4}\right)+\frac{2}{\ln (2)^{2} \sigma^{4}}\left\|\boldsymbol{\xi}_{i}\right\|^{4},
\end{aligned}
$$

where $(a)(b)$ follow because for $K_{1}, K_{2} \geq 0,\left(K_{1}+K_{2}\right)^{2} \leq 2 K_{1}^{2}+2 K_{2}^{2}$, $(c)$ follows because $\left\|\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1}\right\|=$ $\frac{1}{\lambda_{\min }\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)} \leq 1,(d)$ follows because $\mathbf{A} \preceq\|\mathbf{A}\| \mathbf{I}_{n}$ for any Hermitian $\mathbf{A} \in \mathbb{C}^{n \times n}$ (by Lemma 10 in the Appendix), (e) follows because $\ln \operatorname{det}\left(\mathbf{I}_{n}+\mathbf{A}\right) \leq \operatorname{tr}(\mathbf{A})$ for $\mathbf{A}$ positive semi-definite and because for $K_{1}, K_{2} \geq 0,\left(K_{1}+K_{2}\right)^{2} \leq 2 K_{1}^{2}+2 K_{2}^{2}$ and $(f)$ follows because $\|\tilde{\mathbf{Q}}\|=\lambda_{\max }(\tilde{\mathbf{Q}}) \leq \operatorname{tr}(\tilde{\mathbf{Q}}) \leq P$. This implies using the fact that $\mathbf{G}_{i}$ and $\boldsymbol{T}_{i}$ are independent that

$$
\begin{aligned}
\mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right] & \leq \frac{4 P^{2}}{\ln (2)^{2} \sigma^{4}} N_{R}^{2} \mathbb{E}\left[\left\|\mathbf{G}_{i}\right\|^{4}\right]+\frac{32}{\ln (2)^{2} \sigma^{4}}\left(\mathbb{E}\left[\left\|\mathbf{G}_{i}\right\|^{4}\right] \mathbb{E}\left[\left\|\boldsymbol{T}_{i}\right\|^{4}\right]+\mathbb{E}\left[\left\|\boldsymbol{\xi}_{i}\right\|^{4}\right]\right)+\frac{2}{\ln (2)^{2} \sigma^{4}} \mathbb{E}\left[\left\|\boldsymbol{\xi}_{i}\right\|^{4}\right] \\
& \leq \frac{4 P^{2}}{\ln (2)^{2} \sigma^{4}} N_{R}^{2} c_{1}+\frac{16}{\ln (2)^{2} \sigma^{4}}\left(c_{1} c_{2}+c_{3}\right)+\frac{2}{\ln (2)^{2} \sigma^{4}} c_{3} \\
& =c^{\prime \prime}
\end{aligned}
$$

for some $c_{1}, c_{2}, c_{3}>0$, where we used that $\mathbb{E}\left[\left\|\mathbf{G}_{i}\right\|^{4}\right]$ is bounded from above (by Lemma 13 in the Appendix) and that $\mathbb{E}\left[\left\|\boldsymbol{T}_{i}\right\|^{4}\right]$ and $\mathbb{E}\left[\left\|\boldsymbol{\xi}_{i}\right\|^{4}\right]$ are both bounded from above (by Lemma 18 in the Appendix) and where $c^{\prime \prime}>0$.

As a result, we have

$$
\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right] \leq \frac{c^{\prime \prime}}{n}
$$

To summarize, we have proved that

- $\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} m(i, k)+\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k) \leq \frac{2 c^{\prime}}{n(1-\sqrt{\alpha})}$
- $\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right] \leq \frac{c^{\prime \prime}}{n}$

Now, from (18), we know that

$$
\begin{aligned}
& \operatorname{var}\left(\frac{i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)}{n}\right) \\
& \leq \frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=1}^{i-1} m(i, k)+\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k)+\frac{1}{n^{2}} \sum_{i=1}^{n} \mathbb{E}\left[i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right)^{2}\right]
\end{aligned}
$$

To conclude, it follows that

$$
\begin{aligned}
\operatorname{var}\left(\frac{i\left(\boldsymbol{T}^{n} ; \boldsymbol{Z}^{n}, \mathbf{G}^{n}\right)}{n}\right) & \leq \frac{2 c^{\prime}}{n(1-\sqrt{\alpha})}+\frac{c^{\prime \prime}}{n} \\
& =\kappa(n)
\end{aligned}
$$

where $\lim _{n \rightarrow \infty} \kappa(n)=0$. This completes the proof of Lemma 4 .

## V. Conclusion

In this paper, we studied the problem of message transmission over time-varying MIMO first-order Gauss-Markov Rayleigh fading channels with average power constraint and with CSIR, as an example of infinite-state Markov fading channels. The novelty of our work lies in establishing a single-letter characterization of the channel capacity. As a future work, it would be interesting to study the capacity of time-varying MIMO Rayleigh fading channels when a higher-order Gauss-Markov model is used to describe the channel variations over the time.

## APPENDIX

## A. Auxiliary Lemmas

Lemma 10. For any Hermitian matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$, the matrix

$$
\|\mathbf{A}\| \mathbf{I}_{n}-\mathbf{A}
$$

is positive semi-definite.
Proof. Since $\mathbf{A}$ is Hermitian, we know that for any $\boldsymbol{x} \in \mathbb{C}^{n}, \boldsymbol{x}^{H} \mathbf{A} \boldsymbol{x}$ is real. Therefore, for any $\boldsymbol{x} \in \mathbb{C}^{n} \backslash\{\boldsymbol{0}\}$,

$$
\begin{aligned}
\boldsymbol{x}^{H} \mathbf{A} \boldsymbol{x} & \leq\left|\boldsymbol{x}^{H} \mathbf{A} \boldsymbol{x}\right| \\
& \leq\|\mathbf{A}\|\|\boldsymbol{x}\|^{2} .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\boldsymbol{x}^{H}\left(\|\mathbf{A}\| \mathbf{I}_{n}-\mathbf{A}\right) \boldsymbol{x} & =\boldsymbol{x}^{H}\|\mathbf{A}\| \mathbf{I}_{n} \boldsymbol{x}-\boldsymbol{x}^{H} \mathbf{A} \boldsymbol{x} \\
& =\|\mathbf{A}\|\|\boldsymbol{x}\|^{2}-\boldsymbol{x}^{H} \mathbf{A} \boldsymbol{x} \\
& \geq 0
\end{aligned}
$$

Lemma 11. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be any positive-definite Hermitian matrix with $\lambda_{\min }(\mathbf{A})$ being its smallest eigenvalue and let $\mathbf{B} \in \mathbb{C}^{n \times n}$ be any positive semi-definite matrix, then

$$
\log \operatorname{det}(\mathbf{A}+\mathbf{B}) \leq \log \operatorname{det}(\mathbf{A})+\log \operatorname{det}\left(\mathbf{I}_{n}+\frac{1}{\lambda_{\min }(\mathbf{A})} \mathbf{B}\right)
$$

Proof.

$$
\begin{aligned}
\operatorname{det}(\mathbf{A}+\mathbf{B}) & =\operatorname{det}(\mathbf{A}) \operatorname{det}\left(\mathbf{I}_{n}+\mathbf{A}^{-1} \mathbf{B}\right) \\
& =\operatorname{det}(\mathbf{A}) \operatorname{det}\left(\mathbf{I}_{n}+\mathbf{B A}^{-1}\right) \\
& =\operatorname{det}(\mathbf{A}) \operatorname{det}\left(\mathbf{I}_{n}+\mathbf{B}^{\frac{1}{2}} \mathbf{B}^{\frac{1}{2}} \mathbf{A}^{-1}\right) \\
& =\operatorname{det}(\mathbf{A}) \operatorname{det}\left(\mathbf{I}_{n}+\mathbf{B}^{\frac{1}{2}} \mathbf{A}^{-1} \mathbf{B}^{\frac{1}{2}}\right) \\
& \stackrel{(a)}{\leq} \operatorname{det}(\mathbf{A}) \operatorname{det}\left(\mathbf{I}_{n}+\mathbf{B}^{\frac{1}{2}} \frac{1}{\lambda_{\min }(\mathbf{A})} \mathbf{I}_{n} \mathbf{B}^{\frac{1}{2}}\right) \\
& =\operatorname{det}(\mathbf{A}) \operatorname{det}\left(\mathbf{I}_{n}+\frac{1}{\lambda_{\min }(\mathbf{A})} \mathbf{I}_{n} \mathbf{B}\right) \\
& =\operatorname{det}(\mathbf{A}) \operatorname{det}\left(\mathbf{I}_{n}+\frac{1}{\lambda_{\min }(\mathbf{A})} \mathbf{B}\right)
\end{aligned}
$$

where (a) follows from the following properties:

1) For any positive semi-definite Hermitian matrices $\mathbf{M}_{1}$ and $\mathbf{M}_{2}$, if $\mathbf{M}_{1}-\mathbf{M}_{2}$ is Hermitian positive semi-definite then $\operatorname{det}\left(\mathbf{M}_{1}\right) \geq \operatorname{det}\left(\mathbf{M}_{2}\right)$.
2) For any positive definite Hermitian matrix $\mathbf{M} \in \mathbb{C}^{n \times n}$, with minimum eigenvalue $\lambda_{\text {min }}(\mathbf{M})$, it holds that

$$
\mathbf{M}-\lambda_{\min }(\mathbf{M}) \mathbf{I}
$$

is positive semi-definite,
3) For any positive definite Hermitian matrices $\mathbf{M}$ and $\tilde{\mathbf{M}}$, if $\mathbf{M}-\tilde{\mathbf{M}}$ is positive semi-definite then $\tilde{\mathbf{M}}^{-1}-\mathbf{M}^{-1}$ is positive semi-definite.
Therefore, it follows that

$$
\log \operatorname{det}(\mathbf{A}+\mathbf{B}) \leq \log \operatorname{det}(\mathbf{A})+\log \operatorname{det}\left(\mathbf{I}_{n}+\frac{1}{\lambda_{\min }(\mathbf{A})} \mathbf{B}\right)
$$

Lemma 12. $\mathbb{E}\left[\Lambda\left(\mathbf{G}_{i_{1}}, \mathbf{S}, \tilde{\mathbf{G}}, \tilde{\mathbf{W}}\right)\right] \leq c$ for some $c>0$.
Proof. Recall that

$$
\Lambda\left(\mathbf{G}_{i_{1}}, \mathbf{S}, \tilde{\mathbf{G}}, \tilde{\mathbf{W}}\right)=\left\|\mathbf{G}_{i_{1}}\right\|^{2}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right)
$$

where

$$
\tilde{\mathbf{W}}=\mathbf{S}+\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}}
$$

with $i_{1}<i_{2}$, where $\mathbf{S}=\sqrt{1-\alpha} \sum_{j=i_{1}+1}^{i_{2}} \sqrt{\alpha}{ }^{i_{2}-j} \mathbf{W}_{j}$, and where $\tilde{\mathbf{G}}$ is a random matrix with i.i.d. entries, independent of $\mathbf{G}_{1}$ and $\mathbf{W}_{i}, i=2, \ldots n$ such that $\operatorname{vec}(\mathbf{G}) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)$.

We have

$$
\begin{aligned}
& \mathbb{E}\left[\Lambda\left(\mathbf{G}_{i_{1}}, \mathbf{S}, \tilde{\mathbf{G}}, \tilde{\mathbf{W}}\right)\right] \\
& =\mathbb{E}\left[\left\|\mathbf{G}_{i_{1}}\right\|^{2}\left(P\left\|\mathbf{G}_{i_{1}}\right\|^{2}+P\|\tilde{\mathbf{G}}\|^{2}+2\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+2 P\left\|\mathbf{G}_{i_{1}}\right\|\|\mathbf{S}\|\right)\right] \\
& =P \mathbb{E}\left[\left\|\mathbf{G}_{i_{1}}\right\|^{4}\right]+P \mathbb{E}\left[\left\|\mathbf{G}_{i_{1}}\right\|^{2}\|\tilde{\mathbf{G}}\|^{2}\right]+2 \mathbb{E}\left[\left\|\mathbf{G}_{i_{1}}\right\|^{2} \| \tilde{\mathbf{W}} \tilde{\mathbf{Q}}^{\left.\tilde{\mathbf{G}}^{H} \|\right]+2 P \mathbb{E}\left[\left\|\mathbf{G}_{i_{1}}\right\|^{3}\|\mathbf{S}\|\right]}\right. \\
& =P \mathbb{E}\left[\left\|\mathbf{G}_{i_{1}}\right\|^{4}\right]+P \mathbb{E}\left[\left\|\mathbf{G}_{i_{1}}\right\|^{2}\right] \mathbb{E}\left[\|\tilde{\mathbf{G}}\|^{2}\right]+2 \mathbb{E}\left[\left\|\mathbf{G}_{i_{1}}\right\|^{2}\right] \mathbb{E}\left[\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|\right]+2 P \mathbb{E}\left[\left\|\mathbf{G}_{i_{1}}\right\|^{3}\right] \mathbb{E}[\|\mathbf{S}\|] \\
& =P \mathbb{E}\left[\|\tilde{\mathbf{G}}\|^{4}\right]+P \mathbb{E}\left[\|\tilde{\mathbf{G}}\|^{2}\right]^{2}+2 \mathbb{E}\left[\|\tilde{\mathbf{G}}\|^{2}\right] \mathbb{E}\left[\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|\right]+2 P \mathbb{E}\left[\|\tilde{\mathbf{G}}\|^{3}\right] \mathbb{E}[\|\mathbf{S}\|],
\end{aligned}
$$

where we used that $\mathbf{G}_{i_{1}}$ is independent of $(\tilde{\mathbf{W}}, \tilde{\mathbf{G}})$ and that $\tilde{\mathbf{G}}$ has the same distribution as $\mathbf{G}_{i_{1}}$.
By Lemma 13, we know that $\mathbb{E}\left[\|\tilde{\mathbf{G}}\|^{\ell}\right]<\infty$ for all integers $\ell$. Therefore, to complete the proof, we have to show that $\mathbb{E}\left[\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|\right]$ and $\mathbb{E}[\|\mathbf{S}\|]$ are both bounded from above.

It holds that

$$
\begin{aligned}
& \mathbb{E}\left[\left\|\tilde{\mathbf{W}} \tilde{\mathbf{Q}}^{H} \tilde{\mathbf{G}}^{H}\right\|\right] \\
& =\mathbb{E}\left[\left\|\left(\mathbf{S}+\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}}\right) \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|\right] \\
& =\mathbb{E}\left[\left\|\mathbf{S} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}+\sqrt{\alpha}^{i_{2}-i_{1}} \tilde{\mathbf{G}} \tilde{\mathbf{Q}}^{\mathbf{G}^{H}}\right\|\right] \\
& \leq \mathbb{E}\left[\left\|\mathbf{S} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|+\sqrt{\alpha}^{i_{2}-i_{1}}\left\|\tilde{\mathbf{G}} \tilde{\mathbf{Q}} \tilde{\mathbf{G}}^{H}\right\|\right] \\
& \leq\|\tilde{\mathbf{Q}}\| \mathbb{E}\left[\|\mathbf{S}\|\|\tilde{\mathbf{G}}\|+\|\tilde{\mathbf{G}}\|^{2}\right] \\
& \leq P \mathbb{E}\left[\|\mathbf{S}\|\|\tilde{\mathbf{G}}\|+\|\tilde{\mathbf{G}}\|^{2}\right] \\
& =P\left(\mathbb{E}[\|\mathbf{S}\|] \mathbb{E}[\|\tilde{\mathbf{G}}\|]+\mathbb{E}\left[\|\tilde{\mathbf{G}}\|^{2}\right]\right),
\end{aligned}
$$

where we used that $\tilde{\mathbf{G}}$ and $\mathbf{S}$ are independent in the last step, since $\tilde{\mathbf{G}}$ and $\mathbf{W}_{i_{1}+1}, \ldots \mathbf{W}_{i_{2}}$ are independent.

Therefore, to complete the proof, it suffices to show that $\mathbb{E}[\|\mathbf{S}\|]$ is bounded from above.
We have

$$
\begin{aligned}
& \mathbb{E}[\|\mathbf{S}\|] \\
& \left.=\mathbb{E}\left[\left\|\sqrt{1-\alpha} \sum_{j=i_{1}+1}^{i_{2}} \sqrt{\alpha}^{i_{2}-j} \mathbf{W}_{j}\right\|\right]\right] \\
& \leq \mathbb{E}\left[\sqrt{1-\alpha} \sum_{j=i_{1}+1}^{i_{2}} \sqrt{\alpha}^{i_{2}-j}\left\|\mathbf{W}_{j}\right\|\right] \\
& =\sqrt{1-\alpha} \sum_{j=i_{1}+1}^{i_{2}} \sqrt{\alpha}^{i_{2}-j} \mathbb{E}\left[\left\|\mathbf{W}_{j}\right\|\right] \\
& =\sqrt{1-\alpha} \mathbb{E}[\|\tilde{\mathbf{G}}\|] \sum_{j=i_{1}+1}^{i_{2}} \sqrt{\alpha}^{i_{2}-j} \\
& =\frac{\sqrt{1-\alpha}}{1-\sqrt{\alpha}}\left(1-\sqrt{\alpha}^{i_{2}-i_{1}}\right) \mathbb{E}[\|\tilde{\mathbf{G}}\|] \\
& \leq \frac{\sqrt{1-\alpha}}{1-\sqrt{\alpha}} \mathbb{E}[\|\tilde{\mathbf{G}}\|],
\end{aligned}
$$

where we used that

$$
\begin{aligned}
\sum_{j=i_{1}+1}^{i_{2}} \sqrt{\alpha}^{i_{2}-j} & =\sqrt{\alpha}^{i_{2}} \sum_{j=i_{1}+1}^{i_{2}}\left(\frac{1}{\sqrt{\alpha}}\right)^{j} \\
& =\sqrt{\alpha}^{i_{2}}\left(\frac{1}{\sqrt{\alpha}}\right)^{i_{1}+1} \frac{1-\left(\frac{1}{\sqrt{\alpha}}\right)^{i_{2}-i_{1}}}{1-\frac{1}{\sqrt{\alpha}}} \\
& =\frac{\sqrt{\alpha}^{i_{2}-i_{1}}-1}{\sqrt{\alpha}-1} \\
& =\frac{1-\sqrt{\alpha}^{i_{2}-i_{1}}}{1-\sqrt{\alpha}}
\end{aligned}
$$

and that $\tilde{\mathbf{G}}$ has the same distribution as each of the $\mathbf{W}_{i}$. Therefore, $\mathbb{E}[\|\mathbf{S}\|]$ is bounded from above. This proves that $\mathbb{E}\left[\Lambda\left(\mathbf{G}_{i_{1}}, \mathbf{S}, \tilde{\mathbf{G}}, \tilde{\mathbf{W}}\right)\right] \leq c$ for some $c>0$.

Lemma 13. Let $\mathbf{G} \in \mathbb{C}^{N_{R} \times N_{T}}$ a random matrix with i.i.d. entries such that

$$
\operatorname{vec}(\mathbf{G}) \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R} N_{T}}, \mathbf{I}_{N_{R} N_{T}}\right)
$$

Then for all integers $\ell \geq 0$, it holds that $\mathbb{E}\left[\|\mathbf{G}\|^{\ell}\right]<\infty$.
Proof. we will use the $\epsilon$-net argument.
Definition 4. [21] Let $(T, d)$ be a metric space. Let $\mathcal{K} \subset T$. Let $\epsilon>0$. A subset $\mathcal{N} \subseteq \mathcal{K}$ is called an $\epsilon$-net of $\mathcal{K}$ if every point in $\mathcal{K}$ is within distance $\epsilon$ of some point of $\mathcal{N}$, i.e

$$
\forall \boldsymbol{x} \in \mathcal{K} \exists \boldsymbol{x}_{0} \in \mathcal{N}: d\left(\boldsymbol{x}, \boldsymbol{x}_{0}\right) \leq \epsilon
$$

Definition 5. [21] The smallest possible cardinality of an $\epsilon$-net of $\mathcal{K}$ is called the covering number of $\mathcal{K}$ and is denoted by $\mathcal{N}(k, d, \epsilon)$.

Let $\epsilon \in\left(0, \frac{1}{2}\right)$. It holds that

$$
\|\mathbf{G}\| \leq\left\|\mathbf{G}_{R}\right\|+\left\|\mathbf{G}_{I}\right\|
$$

where $\mathbf{G}_{R}=\operatorname{Re}(\mathbf{G}) \in \mathbb{R}^{N_{R} \times N_{T}}$ and $\mathbf{G}_{I}=\operatorname{Im}(\mathbf{G}) \in \mathbb{R}^{N_{R} \times N_{T}}$ contain the real and imaginary parts of the matrix $\mathbf{G}$, respectively. Therefore, it follows that

$$
\begin{aligned}
\|\mathbf{G}\|^{\ell} & \leq\left(\left\|\mathbf{G}_{R}\right\|+\left\|\mathbf{G}_{I}\right\|\right)^{\ell} \\
& \leq 2^{\ell-1}\left(\left\|\mathbf{G}_{R}\right\|^{\ell}+\left\|\mathbf{G}_{I}\right\|^{\ell}\right)
\end{aligned}
$$

where we used that for any integer $\ell$, and for any positive real numbers $a$ and $b$, we have $(a+b)^{\ell} \leq 2^{\ell-1}\left(a^{\ell}+b^{\ell}\right)$ (see Lemma 14 below). This yields

$$
\begin{equation*}
\mathbb{E}\left[\|\mathbf{G}\|^{\ell}\right] \leq 2^{\ell-1}\left(\mathbb{E}\left[\left\|\mathbf{G}_{R}\right\|^{\ell}\right]+\left[\left\|\mathbf{G}_{I}\right\|^{\ell}\right]\right) \tag{35}
\end{equation*}
$$

It has been shown in [21] that the covering number for the unit Euclidean sphere $\mathcal{S}^{n-1}$ satisfies for $\epsilon>0$ the following:

$$
\begin{equation*}
\mathcal{N}\left(S^{n-1}, \epsilon\right) \leq\left(\frac{2}{\epsilon}+1\right)^{n} \tag{36}
\end{equation*}
$$

Furthermore, it has been shown in [21] that for any real matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and any $\epsilon \in\left(0, \frac{1}{2}\right)$, for any $\epsilon$-net $\mathcal{N}$ of the sphere $\mathcal{S}^{n-1}$ and any $\epsilon$-net $\mathcal{M}$ of the sphere $\mathcal{S}^{m-1}$, it holds that

$$
\|\mathbf{A}\| \leq \frac{1}{1-2 \epsilon_{\boldsymbol{x} \in \mathcal{N}, \boldsymbol{y} \in \mathcal{M}}} \sup _{\boldsymbol{A}}\langle\mathbf{A} \boldsymbol{x}, \boldsymbol{y}\rangle .
$$

Let $\tilde{\mathcal{N}}$ be an $\epsilon$-net of the sphere $S^{N_{T}-1}$ and let $\tilde{\mathcal{M}}$ be an $\epsilon$-net $\tilde{\mathcal{M}}$ of the sphere $S^{N_{R}-1}$, both with the smallest possible cardinality. It follows for $\epsilon \in\left(0, \frac{1}{2}\right)$ that

$$
\left\|\mathbf{G}_{R}\right\|^{\ell} \leq\left(\frac{1}{1-2 \epsilon}\right)^{\ell}\left(\sup _{\boldsymbol{t} \in \tilde{\mathcal{N}}, \boldsymbol{z} \in \tilde{\mathcal{M}}}\left\langle\mathbf{G}_{R} \boldsymbol{t}, \boldsymbol{z}\right\rangle\right)^{\ell}
$$

and

$$
\left\|\mathbf{G}_{I}\right\|^{\ell} \leq\left(\frac{1}{1-2 \epsilon}\right)^{\ell}\left(\sup _{\boldsymbol{t} \in \tilde{\mathcal{N}}, \boldsymbol{z} \in \tilde{\mathcal{M}}}\left\langle\mathbf{G}_{I} \boldsymbol{t}, \boldsymbol{z}\right\rangle\right)^{\ell}
$$

Furthermore, it follows from (36) for $\epsilon \in\left(0, \frac{1}{2}\right)$ that

$$
|\tilde{\mathcal{N}}| \leq\left(\frac{2}{\epsilon}+1\right)^{N_{T}}=c_{1}
$$

and that

$$
|\tilde{\mathcal{M}}| \leq\left(\frac{2}{\epsilon}+1\right)^{N_{R}}=c_{2}
$$

for some $c_{1}, c_{2}>0$. We have for $\epsilon \in\left(0, \frac{1}{2}\right)$

$$
\begin{aligned}
& \mathbb{E}\left[\left\|\mathbf{G}_{R}\right\|^{\ell}\right] \\
& \leq\left(\frac{1}{1-2 \epsilon}\right)^{\ell} \mathbb{E}\left[\left(\sup _{\boldsymbol{t} \in \tilde{\mathcal{N}}, \boldsymbol{z} \in \tilde{\mathcal{M}}}\left\langle\mathbf{G}_{R} \boldsymbol{t}, \boldsymbol{z}\right\rangle\right)^{\ell}\right] \\
& \leq\left(\frac{1}{1-2 \epsilon}\right)^{\ell} \mathbb{E}\left[\left(\sum_{\boldsymbol{t} \in \tilde{\mathcal{N}} \boldsymbol{z} \in \tilde{\mathcal{M}}} \sum_{j=1}^{N_{R}} \sum_{i=1}^{N_{T}}\left(\mathbf{G}_{R}\right)_{j i} \boldsymbol{t}_{i} \boldsymbol{z}_{j}\right)^{\ell}\right] \\
& \leq \frac{\left(c_{1} c_{2}\right)^{\ell}}{(1-2 \epsilon)^{\ell}} \mathbb{E}\left[\left(\sum_{j=1}^{N_{R}} \sum_{i=1}^{N_{T}}\left(\mathbf{G}_{R}\right)_{j i}\right)^{\ell}\right] \\
& <\infty
\end{aligned}
$$

where we used that $S_{R}=\left(\sum_{j=1}^{N_{R}} \sum_{i=1}^{N_{T}}\left(\mathbf{G}_{R}\right)_{j i}\right)$ is the sum of independent and identically distributed Gaussian random variables with mean 0 and variance $\frac{1}{2}$. Therefore $S_{R}$ is a Gaussian random variable with mean 0 and variance $\frac{N_{R} N_{T}}{2}$. Therefore the $\ell^{\text {th }}$ moment of $S_{R}$ is finite.

Analogously, one can show that $\mathbb{E}\left[\left\|\mathbf{G}_{I}\right\|^{\ell}\right]<\infty$.
Thus, we can conclude using (35) that $\mathbb{E}\left[\|\mathbf{G}\|^{\ell}\right]<\infty$.

Lemma 14. For any real numbers $a, b$ and for any integer $\ell \geq 0$.

$$
|a+b|^{\ell} \leq 2^{\ell-1}\left(|a|^{\ell}+|b|^{\ell}\right)
$$

Proof. The statement of the lemma is clear for $\ell=0$. Now for any integer $\ell \geq 1$, the function $\Psi(x)=x^{\ell}$ is convex for $x \geq 0$, since its second derivative is equal to $\ell(\ell-1) x^{\ell-2} \geq 0$.

Therefore, by using the convexity of $\Psi$, it follows that

$$
\begin{aligned}
\left|\frac{a+b}{2}\right|^{\ell} & \leq\left(\frac{|a|+|b|}{2}\right)^{\ell} \\
& \leq \frac{|a|^{p}+|b|^{\ell}}{2}
\end{aligned}
$$

Lemma 15. For any $0<\alpha<1$, it holds that

$$
\sum_{i=1}^{n} \sum_{k=1}^{i-1} \alpha^{i-k} \leq \frac{n}{1-\alpha}
$$

Proof. We have

$$
\begin{aligned}
& \sum_{i=1}^{n} \sum_{k=1}^{i-1} \alpha^{i-k} \\
& =\sum_{i=1}^{n} \alpha^{i} \sum_{k=1}^{i-1}\left(\frac{1}{\alpha}\right)^{k} \\
& =\sum_{i=1}^{n} \alpha^{i} \frac{1}{\alpha} \frac{1-\left(\frac{1}{\alpha}\right)^{i-1}}{1-\frac{1}{\alpha}} \\
& =\sum_{i=1}^{n} \alpha^{i} \frac{1-\left(\frac{1}{\alpha}\right)^{i-1}}{\alpha-1} \\
& =\sum_{i=1}^{n} \frac{\alpha^{i}-\alpha}{\alpha-1} \\
& =\sum_{i=1}^{n} \frac{\alpha-\alpha^{i}}{1-\alpha} \\
& =\frac{n \alpha}{1-\alpha}-\sum_{i=1}^{n} \frac{\alpha^{i}}{1-\alpha} \\
& \leq \frac{n \alpha}{1-\alpha} \\
& \leq \frac{n}{1-\alpha}
\end{aligned}
$$

Lemma 16. For any $0<\alpha<1$ it holds that

$$
\sum_{i=1}^{n} \sum_{k=i+1}^{n} \alpha^{k-i} \leq \frac{n}{1-\alpha}
$$

Proof. We have

$$
\begin{aligned}
& \sum_{i=1}^{n} \sum_{k=i+1}^{n} \alpha^{k-i} \\
& =\sum_{i=1}^{n}\left(\frac{1}{\alpha}\right)^{i} \sum_{k=i+1}^{n} \alpha^{k} \\
& =\sum_{i=1}^{n}\left(\frac{1}{\alpha}\right)^{i} \alpha^{i+1} \frac{\left(1-\alpha^{n-i}\right)}{1-\alpha} \\
& =\sum_{i=1}^{n} \frac{\alpha\left(1-\alpha^{n-i}\right)}{1-\alpha} \\
& =\frac{n \alpha}{1-\alpha}-\frac{\alpha^{n+1}}{1-\alpha} \sum_{i=1}^{n}\left(\frac{1}{\alpha}\right)^{i} \\
& \leq \frac{n \alpha}{1-\alpha} \\
& \leq \frac{n}{1-\alpha}
\end{aligned}
$$

Lemma 17. $\forall i \in\{1, \ldots, n\}$

$$
\begin{aligned}
& i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) \\
& =\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)-\frac{1}{\ln (2) \sigma^{2}}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)^{H}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)+\frac{1}{\ln (2) \sigma^{2}} \boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1} \boldsymbol{Z}_{i}
\end{aligned}
$$

where $\boldsymbol{T}_{i} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{T}}, \tilde{\mathbf{Q}}\right), i=1 \ldots n$.

Proof. Notice that

$$
\begin{aligned}
i\left(\boldsymbol{T}_{i} ; \boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) & =\log \left(\frac{p_{\boldsymbol{Z}_{i}, \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i}, \mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{p_{\boldsymbol{Z}_{i}, \mathbf{G}_{i}}\left(\boldsymbol{Z}_{i}, \mathbf{G}_{i}\right) p_{\boldsymbol{T}_{i}}\left(\boldsymbol{T}_{i}\right)}\right) \\
& =\log \left(\frac{p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}\right)}\right)
\end{aligned}
$$

where we used that $\boldsymbol{T}_{i}$ and $\mathbf{G}_{i}$ are independent.
It holds that

$$
\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{G}_{i} \boldsymbol{T}_{i}, \sigma^{2} \mathbf{I}_{N_{R}}\right)
$$

and that

$$
\boldsymbol{Z}_{i} \mid \mathbf{G}_{i} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N_{R}}, \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}+\sigma^{2} \mathbf{I}_{N_{R}}\right)
$$

It follows that

$$
\begin{aligned}
& \log \frac{p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}, \boldsymbol{T}_{i}\right)}{p_{\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}}\left(\boldsymbol{Z}_{i} \mid \mathbf{G}_{i}\right)} \\
& =\log \left[\frac{\frac{1}{\pi^{N_{R} \operatorname{det}\left(\sigma^{2} \mathbf{I}_{N_{R}}\right)} \exp \left(\frac{-1}{\sigma^{2}}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)^{H}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)\right)}}{\frac{1}{\pi^{N_{R}} \operatorname{det}\left(\mathbf{G}_{i} \hat{\mathbf{Q}} \mathbf{G}_{i}^{H}+\sigma^{2} \mathbf{I}_{N_{R}}\right)} \exp \left(-\frac{1}{\sigma^{2}} \boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1} \boldsymbol{Z}_{i}\right)}\right] \\
& =\log \left[\frac{\operatorname{det}\left(\mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}+\sigma^{2} \mathbf{I}_{N_{R}}\right)}{\operatorname{det}\left(\sigma^{2} \mathbf{I}_{N_{R}}\right)} 2\left(\frac{-1}{\ln \left(2 \sigma^{2}\right.}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)^{H}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)+\frac{1}{\ln (2) \sigma^{2}} \boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1} \boldsymbol{Z}_{i}\right)\right] \\
& =\log \operatorname{det}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)-\frac{1}{\ln (2) \sigma^{2}}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)^{H}\left(\boldsymbol{Z}_{i}-\mathbf{G}_{i} \boldsymbol{T}_{i}\right)+\frac{1}{\ln (2) \sigma^{2}} \boldsymbol{Z}_{i}^{H}\left(\mathbf{I}_{N_{R}}+\frac{1}{\sigma^{2}} \mathbf{G}_{i} \tilde{\mathbf{Q}} \mathbf{G}_{i}^{H}\right)^{-1} \boldsymbol{Z}_{i} .
\end{aligned}
$$

Lemma 18. For any random vector $\boldsymbol{X}=\left(X_{1}, \ldots, X_{N}\right)^{T} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}_{N}, \mathbf{O}\right)$ with $\operatorname{tr}(\mathbf{O}) \leq \nu, \nu>0, \mathbb{E}\left[\|\boldsymbol{X}\|^{4}\right]$ is bounded from above.

Proof. It holds that

$$
\begin{aligned}
\|\boldsymbol{X}\|^{4} & =\left(\sum_{\ell=1}^{N}\left|X_{\ell}\right|^{2}\right)\left(\sum_{\ell=1}^{N}\left|X_{\ell}\right|^{2}\right) \\
& =\sum_{\ell=1}^{N} \sum_{s=1, s \neq \ell}^{N}\left|X_{\ell}\right|^{2}\left|X_{s}\right|^{2}+\sum_{\ell=1}^{N}\left|X_{\ell}\right|^{4} .
\end{aligned}
$$

This yields

$$
\begin{aligned}
\mathbb{E}\left[\|\boldsymbol{X}\|^{4}\right] & =\sum_{\ell=1}^{N} \sum_{s=1, s \neq \ell}^{N} \mathbb{E}\left[\left|X_{\ell}\right|^{2}\left|X_{s}\right|^{2}\right]+\sum_{\ell=1}^{N} \mathbb{E}\left[\left|X_{\ell}\right|^{4}\right] \\
& \leq \sum_{\ell=1}^{N} \sum_{s=1, s \neq \ell}^{N} \sqrt{\mathbb{E}\left[\left|X_{\ell}\right|^{4}\right] \mathbb{E}\left[\left|X_{s}\right|^{4}\right]}+\sum_{\ell=1}^{N} \mathbb{E}\left[\left|X_{\ell}\right|^{4}\right]
\end{aligned}
$$

where we used Cauchy Schwarz's inequality. Since $\operatorname{tr}(\mathbf{O}) \leq \nu$, it follows that for all $\ell=1, \ldots, N$

$$
X_{\ell} \sim \mathcal{N}_{\mathbb{C}}\left(0, v_{\ell}\right)
$$

where $v_{\ell} \leq \nu$. Therefore, $\mathbb{E}\left[\left|X_{\ell}\right|^{4}\right], \ell=1 \ldots N$, is bounded from above and so is $\mathbb{E}\left[\|\boldsymbol{X}\|^{4}\right]$.

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